ETH ZÜRICH

MASTER THESIS

Sup-norms of Modular Forms of real and half-integral Weight

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1 Abstract

In this thesis we study different approaches to bound the supremum of $y^{\frac{k}{2}}|f(z)|$, where f is a cusp form of arbitrary real weight k with respect to a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$. The techniques involve bounding Fourier coefficients of Poincaré series and bounds on the Bergman kernel. The results are completely uniform in every aspect. We further specialize to the case of Hecke eigenforms of half-integral weight and improve upon the previously given convexity bounds.

2 Introduction

Supremum norms of Maass and holomorphic cusp forms have been studied in various ways. In the case of Maass forms, Iwaniec-Sarnak [IS95] obtained the first non-trivial result in the eigenvalue aspect. They have been further studied in the level aspect by Blomer-Holowinsky [BH10], Templier [Tem10] and Harcos-Templier [HT12], [HT13] to name a few. Templier was able to unify both best results in a hybrid bound [Tem11].
In the case of holomorphic forms, results in the weight aspect have been obtained by Xia [Xia07], Rudnick [Rud05], Blomer-Khan-Young [BKY13] and Friedman-Jorgenson-Kramer [FJK13]. In the level aspect non-trivial results have been given by Blomer-Holowinsky [BH10] and in the case of half integral weight also by Kiral [Kir13].
The question about supremum norms are related to subconvexity of L-functions, the theory of quantum chaos and the mass equidistribution conjecture, which makes it an interesting topic to study.

In this work we study the supremum norm problem for holomorphic cusp forms in a variety of cases, where no non-trivial bounds have been previously written down. In particular we generalize the results obtained in [FJK13] in absolute uniformity to modular forms *of arbitrary real weight* with respect to a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$. In the second part of the thesis we specialize to *half-integral weight* newforms of level 4, breaking the previously given convexity bounds.
To motivate some of our results in Section 3 we recall a result of Rudnick [Rud05], who proved that for a fixed compact subset K of the upper-half plane $\mathbb{H}$ and a cusp form f of weight $k \in 2\mathbb{Z}$ for the full modular group $\mathrm{SL}_2(\mathbb{Z})$ we have $\sup_{z\in K} y^{\frac{k}{2}}|f(z)| \ll_K k^{\frac{1}{2}}\|f\|_2$. This result is essentially the best possible as there is a family of modular forms, which admit their supremum in a compact set and satisfy $\sup_{z\in K} y^{\frac{k}{2}}|f(z)| \gg k^{\frac{1}{2}-\epsilon}\|f\|_2$.
We generalize this result of Rudnick uniformly to arbitrary real weight k, finite index subgroup Γ and automorphy factor ν as follows (Theorem 3.23):

$$\sup_{z\in K} y^{\frac{k}{2}}|f(z)| \ll_K [\mathrm{SL}_2(\mathbb{Z}) : \Gamma]^{\frac{1}{2}} \cdot k^{\frac{1}{2}}\|f\|_2.$$

If we do not restrict ourselves to compact sets the situation is different. In this case we are able to show (Theorem 3.24):

$$\sup_{z\in \mathbb{H}} y^{\frac{k}{2}}|f(z)| \ll C(\Gamma, k) \cdot \frac{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]^{\frac{1}{2}} k^{\frac{3}{4}}}{\min\limits_{\tau\in \mathrm{SL}_2(\mathbb{Z})} \kappa_\tau^{\frac{1}{2}}} \cdot \|f\|_2,$$

where $C(\Gamma, k)$ is an explicit small constant and the $\kappa_\tau \in (0, 1]$ are the cusp parameters[1]. $C(\Gamma, k)$ can be taken to be 1 uniformly for all Γ such that $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma] \ll k^{1-\delta}$. This result follows from our generalization of a result by Friedman-Jorgenson-Kramer [FJK13]. They

[1] In the classical setting Γ a congruence subgroup and automorphy factor j^k, where k is an even integer, all cusp parameters are 1.

proved for $k \in 2\mathbb{Z}$: $\sup_{z\in\mathbb{H}} \sum_j y^k |f_j(z)|^2 \ll [\mathrm{SL}_2(\mathbb{Z}) : \Gamma] \cdot k^{\frac{3}{2}}$, where $\{f_j\}$ is an orthonormal basis. If we assume once again $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma] \ll k^{1-\delta}$ and k large enough our generalization can be stated as follows:

$$\sup_{z\in\mathbb{H}} \sum_j y^k |f_j(z)|^2 \asymp \frac{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma] k^{\frac{3}{2}}}{\min\limits_{\tau \in \mathrm{SL}_2(\mathbb{Z})} \kappa_\tau},$$

where k is arbitrary real. We also prove more refined bounds on the quantity $\sum_j y^{\frac{k}{2}} |f_j(z)|$; we refer the reader to Section 3 for details.

So far these results have been without any assumptions on f being eigenforms. For *Hecke eigenforms of integral weight* one can do better as has been shown by Xia [Xia07]. For f a Hecke eigenform on full modular group $\mathrm{SL}_2(\mathbb{Z})$ one has:

$$k^{\frac{1}{4}-\epsilon} \|f\|_2 \ll_\epsilon \sup_{z\in\mathbb{H}} y^{\frac{k}{2}} |f(z)| \ll_\epsilon k^{\frac{1}{4}+\epsilon} \|f\|_2.$$

This begs the question: can one extend Xia's bound to the case of eigenforms of non-integral weight? The theory of Hecke operators of real weight has its difficulties, but it has been well established in the case of half-integral weight. In Section 4 we restrict ourselves to the simplest case of half-integral weight cuspidal eigenforms of level 4 lying in the Kohnen plus space $S^+(\Gamma_0(4), k, j_\Theta^{2k})$. We adopt Xia's method, and succeed in getting the same upper bound as him if we assume the Lindelöf hypothesis (see Theorem 4.8). Unconditionally, using subconvexity results of [MV10] we are able to show (Theorem 4.9)

$$\sup_{z\in\mathbb{H}} y^{\frac{k}{2}} |f(z)| \ll k^{\frac{1}{2}} \|f\|_2.$$

We also obtain some results on lower bounds for the sup-norm (Proposition 4.10, Theorem 4.13).

Some interesting questions that arise are:

- What results can one achieve by this method for half-integral newforms of level $4N$?
- Is it possible to give hybrid bounds on the sup-norm that are subconvex in the weight aspect as well as the level aspect?

To answer these it may also be of use to consider a more general kernel other than the Bergman kernel in order to amplify the contribution of the Hecke eigenform we are interested in and potentially bypass the exponent $\frac{1}{2}$.

3 Modular forms of real weight

In this chapter we follow the notion of modular forms of real weight as defined in [Ran77] and adopt some of the notation used. If not stated otherwise, proofs of claims can be found in [Ran77].

3.1 Definitions and notation

Throughout this thesis we assume that Γ is a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$ with $-I \in \Gamma$ and denote $\hat{\Gamma} := \Gamma / \{\pm I\}$. For $w \in \mathbb{C}$ and $k \in \mathbb{R}$ we let $w^k := \exp(k \cdot \log(w))$, where $\log(w) = \log(|w|) + i \arg(w)$ with $-\pi < \arg(w) \leq \pi$. The symbol $\ll$ denotes the Vinogradov symbol and $f(x) \ll_{A,B,C} g(x)$ means $|f(x)| \leq Kg(x)$, where K depends at most on A, B and C. Further the symbol $f(x) \asymp_A g(x)$ is equivalent to $f(x) \ll_A g(x)$ and $g(x) \ll_A f(x)$.

As usual the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H} = \{z \in \mathbb{C} | \operatorname{Im} z > 0\}$ is given by Möbius transformations:

$$\gamma z = \gamma \cdot z = \frac{az+b}{cz+d}, \quad \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), \forall z \in \mathbb{H}.$$

We extend the same action to the set of cusps $\overline{\mathbb{Q}} = \mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \sqcup \{\infty\}$. The topology on $\mathbb{H} \sqcup \overline{\mathbb{Q}}$ is given in the following way: On $\mathbb{H}$ it is the usual Euclidean topology and a neighborhood basis of ∞ is given by the sets $\infty \cup \{z \in \mathbb{H} | \operatorname{Im} z > M\}$, where $M \in \mathbb{R}^+$. At a cusp $\zeta \in \mathbb{Q}$ a neighborhood basis is given by choosing a $\tau \in \mathrm{SL}_2(\mathbb{Z})$, such that $\tau\zeta = \infty$ and pulling back the neighborhood basis of ∞. The action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H} \sqcup \overline{\mathbb{Q}}$ is then once again properly discontinuous. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ we define

$$j(\gamma, z) = (cz + d), \forall z \in \mathbb{H}.$$

For further convenience we define

$$U := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Definition 3.1. A function $\nu : \Gamma \times \mathbb{H} \to \mathbb{C}$ is called an *automorphy factor* of weight k on Γ if the following conditions are satisfied:

1. $\forall \gamma \in \Gamma : \nu(\gamma, \cdot)$ is a holomorphic function on $\mathbb{H}$,
2. $\forall \gamma \in \Gamma, \forall z \in \mathbb{H} : |\nu(\gamma, z)| = |j(\gamma, z)|^k$,
3. $\forall \gamma, \tau \in \Gamma, \forall z \in \mathbb{H} : \nu(\tau\gamma, z) = \nu(\tau, \gamma z)\nu(\gamma, z)$,
4. $\forall \gamma \in \Gamma, z \in \mathbb{H} : \nu(-\gamma, z) = \nu(\gamma, z)$.

Remark 3.1. For $k \in 2\mathbb{Z}$, j^k defines an automorphy factor of weight k on $\mathrm{SL}_2(\mathbb{Z})$.

Corresponding to ν we can define a *multiplier system* $\upsilon : \Gamma \to S^1$ of weight k on Γ as:

$$\upsilon(\gamma) = \upsilon(\gamma, z) := \frac{\nu(\gamma, z)}{j(\gamma, z)^k}, \quad \forall \gamma \in \Gamma, \forall z \in \mathbb{H}.$$

We remark that the right hand side is indeed independent of z as it is a bounded holomorphic function on $\mathbb{H}$ and thus constant. It satisfies the relation:

$$\upsilon(\tau\gamma) = \sigma(\tau, \gamma)\upsilon(\tau)\upsilon(\gamma), \quad \forall \tau, \gamma \in \Gamma,$$

where

$$\sigma(\tau, \gamma) := \frac{j(\tau, \gamma z)^k j(\gamma, z)^k}{j(\tau\gamma, z)^k}.$$

If ν is an automorphy factor of weight k on Γ and $\tau \in \mathrm{SL}_2(\mathbb{Z})$ we can define a conjugate automorphy factor ν^τ of weight k on $\Gamma^\tau := \tau^{-1}\Gamma\tau$ in the following way:

$$\nu^\tau(\tau^{-1}\gamma\tau, z) = \frac{\nu(\gamma, \tau z) j(\tau, z)^k}{j(\tau, \tau^{-1}\gamma\tau z)^k}, \quad \forall \gamma \in \Gamma, \forall z \in \mathbb{H}.$$

Remark 3.2. If $\tau \in \Gamma$, then $\nu^\tau = \nu$, but $\Gamma^\tau = \Gamma$ does not necessarily imply $\nu^\tau = \nu$, if $\tau \notin \Gamma$.

Definition 3.2. A meromorphic function f on $\mathbb{H}$ is called an *unrestricted modular function* with respect to Γ, k, ν (or υ) if it satisfies

$$f(\gamma z) = \nu(\gamma, z) f(z) = \upsilon(\gamma) j(\gamma, z)^k f(z), \quad \forall \gamma \in \Gamma, z \in \mathbb{H}.$$

If f is a meromorphic function on $\mathbb{H}$, we define the γ-transform $f|_k\gamma$ of f as:

$$(f|_k\gamma)(z) = j(\gamma, z)^{-k} f(\gamma z).$$

In order to define modular functions and forms, we need some conditions at the cusps. For this matter we need to introduce two quantities n_τ and κ_τ. Let $\zeta = \tau\infty \in \overline{\mathbb{Q}}$, where $\tau \in \mathrm{SL}_2(\mathbb{Z})$, be a cusp. Then the stabilizer $\hat{\Gamma}_\zeta$ of ζ is generated by $\tau U^{n_\tau}\tau^{-1}$, where

$$U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

n_τ is called the width of the cusp $\tau\infty$. In the classical setting of $\Gamma = \Gamma_0(N)$ we have that all cusp widths n_τ are integers dividing N.
We also define $0 < \kappa_\tau \leq 1$ to be the real number satisfying:

$$e^{2\pi i \kappa_\tau} = \upsilon^\tau(U^{n_\tau}) = \upsilon^\tau(U^{n_\tau}) j(U^{n_\tau}, z)^k = \nu^\tau(U^{n_\tau}, z) = \upsilon(\tau U^{n_\tau}\tau^{-1}). \tag{1}$$

Classically κ_τ is chosen to be in $[0, 1)$, but in this Section 3 we want κ_τ to be 1 instead of 0 as the quantity κ_τ^{-1} will appear later on.
κ_τ is referred to as the cusp parameter. The quantities n_τ, κ_τ only depend on the equivalence class of the cusp $\tau\infty$ modulo Γ (given ν).

Remark 3.3. n_τ depends on Γ and κ_τ depends on Γ, ν, but for notational purposes they will always be with respect to Γ and ν. This will only cause confusion in one of the proofs, where it is mentioned.

Remark 3.4. In the classical setting Γ a congruence subgroup with automorphy factor $j^{2n}, n \in \mathbb{N}$ the cusp parameter κ_τ is always 1 and the cusp widths are integers dividing N. In case of real weight the cusp parameter is needed to shift the functions such that they admit a Fourier expansion (see Corollary 3.4).

Theorem 3.3. *Let f be an unrestricted modular function with respect to Γ, k, ν. Then we have:*

1. $\forall \tau \in \mathrm{SL}_2(\mathbb{Z}) : f|_k\tau$ *is an unrestricted modular function with respect to* Γ^τ, k, ν^τ,
2. $\forall \tau_1, \tau_2 \in \mathrm{SL}_2(\mathbb{Z}) : f|_k\tau_1\tau_2 = \sigma(\tau_1, \tau_2)(f|_k\tau_1)|_k\tau_2$,
3. $\forall \tau \in \mathrm{SL}_2(\mathbb{Z}), \forall \gamma \in \Gamma : f|_k\gamma\tau = \sigma(\gamma, \tau)\upsilon(\gamma)f|_k\tau$,
4. $\forall \tau \in \mathrm{SL}_2(\mathbb{Z}), \forall z \in \mathbb{H} : (f|_k\tau)(z + n_\tau) = e^{2\pi i \kappa_\tau}(f|_k\tau)(z)$.

Proof. See [Ran77] theorem 4.1.1 page 89. □

Corollary 3.4. *f as in the theorem, then $f|_k\tau$ has a Fourier expansion of the form:*

$$(f|_k\tau)(z) = \sum_{m\in\mathbb{Z}} a_m(\tau) e^{\frac{2\pi i(m+\kappa_\tau)z}{n_\tau}}.$$

Proof. We introduce the function

$$(f|_k\tau)^*(z) := e^{-2\pi i z\kappa_\tau/n_\tau}(f|_k\tau)(z),$$

which satisfies

$$(f|_k\tau)^*(z + n_\tau) = (f|_k\tau)^*(z).$$

And has thus a Fourier expansion of the shape:

$$(f|_k\tau)^*(z) = \sum_{m\in\mathbb{Z}} a_m(\tau) e^{\frac{2\pi i m z}{n_\tau}}.$$

From which the corollary follows. □

Definition 3.5. An unrestricted modular function f with respect to Γ, k, ν is called a *modular function* with respect to Γ, k, ν if for every cusp $\tau\infty$ the Fourier expansion of $(f|_k\tau)^*$ has only finitely many negative terms.

For a modular function we denote the order of f at $\tau\infty$ with respect to Γ as

$$\operatorname{ord}(f,\tau\infty,\Gamma) = \inf\{m \in \mathbb{Z} | a_m(\tau) \neq 0\} + \kappa_\tau,$$

where we use the convention, that $\inf \emptyset = \infty = \sup \mathbb{Z}$. The order is independent of the Γ equivalence class of $\tau\infty$, but the coefficients itself can vary within the equivalence class. For notational convenience we denote by

$$\widehat{(f|_k\tau)}(m) = a_m(\tau)$$

the m-th Fourier coefficient of $(f|_k\tau)^*$. We remark here once again that this definition does not coincide with the classical definition of the Fourier coefficients of modular forms of integral weight. This is due to our choice of the cusp parameter being 1 instead of 0.

Definition 3.6. A modular function f with respect to Γ, k, ν is called a *modular form* with respect to Γ, k, ν if f is holomorphic on $\mathbb{H}$ and for every cusp $\tau\infty$ the order $\operatorname{ord}(f,\tau\infty,\Gamma)$ is non-negative. If for every cusp the order is positive, then f is called a *cusp form* with respect to Γ, k, ν. The space of all modular forms with respect to Γ, k, ν is denoted by $M(\Gamma, k, \nu)$ and the space of all cusp forms with respect to Γ, k, ν is denoted by $S(\Gamma, k, \nu)$.

Theorem 3.7. *The spaces $M(\Gamma, k, \nu)$ and $S(\Gamma, k, \nu)$ are finite dimensional.*

Proof. See [Ran77] theorem 4.2.1 page 102. □

3.2 The Petersson inner product and Poincaré series

Definition 3.8. A *fundamental domain* $\mathbb{F}_\Gamma$ for Γ is a subset of $\mathbb{H}$, which is a finite union of domains, which satisfies:

1. For every $z \in \mathbb{H} : |\mathbb{F}_\Gamma \cap \Gamma z| \leq 1$,
2. For every $z \in \mathbb{H} : \overline{\mathbb{F}_\Gamma} \cap \Gamma z \neq \emptyset$.

Proposition 3.9. *A fundamental domain $\mathbb{F}_I$ for $\mathrm{SL}_2(\mathbb{Z})$ is given by $\mathbb{F}_I = \{z \in \mathbb{H} | |z| > 1 \wedge -\frac{1}{2} < \operatorname{Re} z < \frac{1}{2}\}$. If $\mathrm{SL}_2(\mathbb{Z}) = \bigsqcup \Gamma\tau_i$, then $\bigcup \tau_i \mathbb{F}_I$ is a fundamental domain for Γ.*

Proof. See [Ran77] theorem 2.4.1 page 51. □

Definition 3.10. The Petersson inner product on $S(\Gamma, k, \nu)$ is defined by

$$\langle f, g\rangle_\Gamma = \frac{1}{\mu(\Gamma)} \int_{\mathbb{F}_\Gamma} f(z)\overline{g(z)} y^k \frac{dxdy}{y^2},$$

where $\mu(\Gamma) = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma] = [\mathrm{PSL}_2(\mathbb{Z}) : \hat{\Gamma}]$. It is indeed an inner product and is independent of the choice of the fundamental domain $\mathbb{F}_\Gamma$ and independent of the subgroup Γ, i.e. if $\Gamma' \leq \Gamma$ of finite index, then

$$\langle f, g\rangle_\Gamma = \langle f, g\rangle_{\Gamma'}, \quad \forall f, g \in S(\Gamma, k, \nu) \supseteq S(\Gamma', k, \nu).$$

Definition 3.11. For $k > 2$ we define the m-th Poincaré series of weight k at the cusp $\tau^{-1}\infty$, where $\tau \in \mathrm{SL}_2(\mathbb{Z})$, with respect to Γ, ν as:

$$G_\tau(\Gamma, k, \nu; z, m) = \sum_{\gamma \in \hat{\Gamma}_{\tau^{-1}\infty} \backslash \hat{\Gamma}} \frac{\exp\left(\frac{2\pi i(m+\kappa_{\tau^{-1}})}{n_{\tau^{-1}}}\tau\gamma z\right)}{j(\tau, \gamma z)^k \nu(\gamma, z)}.$$

Theorem 3.12. *The above Poincaré series converges locally uniformly on $\mathbb{H}$ and defines thus a holomorphic function on $\mathbb{H}$ and defines an unrestricted modular form of weight k with respect to Γ, ν.*

1. *They satisfy the relations:*

$$G_{\tau_1}(\Gamma, k, \nu; \cdot, m)|_k \tau_2 = \frac{1}{\sigma(\tau_1, \tau_2)} G_{\tau_1\tau_2}(\Gamma^{\tau_2}, k, \nu^{\tau_2}; \cdot, m);$$

2. *If $m + \kappa_{\tau^{-1}} > 0$, then $G_\tau(\Gamma, k, \nu; \cdot, m) \in S(\Gamma, k, \nu)$;*

3. *If $m+\kappa_{\tau^{-1}} = 0$, then $G_\tau(\Gamma, k, \nu; \cdot, m) \in M(\Gamma, k, \nu)$ non-zero with $\mathrm{ord}(G_\tau, \tau^{-1}\infty, \Gamma) = 0$ and at the other cusps $\zeta \not\equiv \tau^{-1}\infty \bmod \Gamma$ one has $\mathrm{ord}(G_\tau, \zeta, \Gamma) > 0$;*

4. *If $m+\kappa_{\tau^{-1}} < 0$, then $\mathrm{ord}(G_\tau, \tau^{-1}\infty, \Gamma) = m+\kappa$ and at the other cusps $\zeta \not\equiv \tau^{-1}\infty \bmod \Gamma$ one has $\mathrm{ord}(G_\tau, \zeta, \Gamma) > 0$.*

Proof. See [Ran77] theorem 5.1.2 page 136. □

Theorem 3.13. *We have for $k > 2$, $f \in S(\Gamma, k, \nu)$, $\tau \in \mathrm{SL}_2(\mathbb{Z}) \backslash \{-U^l | l \in \mathbb{Z}\}$, $m + \kappa_\tau \geq 0$:*

$$\langle f, G_\tau(\Gamma, k, \nu; \cdot, m)\rangle = \frac{n_{\tau^{-1}}^k \Gamma(k-1)}{\mu(\Gamma)(4\pi(m+\kappa_{\tau^{-1}}))^{k-1}} \widehat{(f|_k\tau^{-1})}(m).$$

Proof. See [Ran77] theorem 5.2.2 page 149. □

Theorem 3.14. *For $\tau \in \mathrm{SL}_2(\mathbb{Z})$ the Poincaré series with $k > 2$ satisfy the following equality:*

$$G_\tau(\Gamma, k, \nu; z, m) = \delta_\tau e^{\frac{2\pi i(m+\kappa_I)z}{n_I}} + \sum_{r+\kappa_I>0} a(r, m; \tau) e^{\frac{2\pi i(r+\kappa_{\tau^{-1}})z}{n_{\tau^{-1}}}},$$

where

$$\delta_\tau = \begin{cases} \frac{e^{\frac{2\pi i s(m+\kappa_I)}{n_I}}}{\upsilon(\tau^{-1}U^s)\sigma(\tau, \tau^{-1})}, & \text{if } \tau^{-1}U^s \in \Gamma, \text{for some } s \in \mathbb{Z}, \\ 0, & \text{else}, \end{cases}$$

and

$$a(r,m;\tau)=\begin{cases}\dfrac{(2\pi)^k}{\Gamma(k)}i^{-k}(r+\kappa_I)^{k-1}\displaystyle\sum_{c=1}^{\infty}\frac{W(\Gamma,\nu;r,m;c)}{(n_I c)^k}, & \textit{if } m+\kappa_{\tau^{-1}}=0,\\ 2\pi i^{-k}\left(\dfrac{n_{\tau^{-1}}}{n_I}\right)^{\frac{k-1}{2}}\left(\dfrac{r+\kappa_I}{m+\kappa_{\tau^{-1}}}\right)^{\frac{k-1}{2}} & \\ \quad\times\displaystyle\sum_{c=1}^{\infty}\frac{W(\Gamma,\nu;r,m;c)}{n_I c}J_{k-1}\left(\frac{4\pi}{c}\sqrt{\frac{(r+\kappa_I)(m+\kappa_{\tau^{-1}})}{n_I n_{\tau^{-1}}}}\right), & \textit{if } m+\kappa_{\tau^{-1}}>0,\\ 2\pi i^{-k}\left(\dfrac{n_{\tau^{-1}}}{n_I}\right)^{\frac{k-1}{2}}\left|\dfrac{r+\kappa_I}{m+\kappa_{\tau^{-1}}}\right|^{\frac{k-1}{2}} & \\ \quad\times\displaystyle\sum_{c=1}^{\infty}\frac{W(\Gamma,\nu;r,m;c)}{n_I c}I_{k-1}\left(\frac{4\pi}{c}\sqrt{\frac{(r+\kappa_I)|m+\kappa_{\tau^{-1}}|}{n_I n_{\tau^{-1}}}}\right), & \textit{if } m+\kappa_{\tau^{-1}}<0.\end{cases}$$

The Bessel functions J_{k-1}, I_{k-1} are given by:

$$J_{k-1}(z)=\sum_{m=0}^{\infty}\frac{(-1)^m(\frac{z}{2})^{2m+k-1}}{\Gamma(m+1)\Gamma(m+k)},$$

$$I_{k-1}(z)=\sum_{m=0}^{\infty}\frac{(\frac{z}{2})^{2m+k-1}}{\Gamma(m+1)\Gamma(m+k)},$$

and the generalized Kloosterman sum is given by:

$$W(\Gamma,\nu;r,m;c)=\sum_{\gamma=\begin{pmatrix}a&*\\c&d\end{pmatrix}\in\tau\mathcal{J}_\tau}\frac{\exp\left(\frac{2\pi i}{c}\left(\frac{(m+\kappa_{\tau^{-1}})a}{n_{\tau^{-1}}}+\frac{(r+\kappa_I)d}{n_I}\right)\right)}{\upsilon(\tau^{-1}\gamma)\sigma(\tau,\tau^{-1})}\sigma(\tau^{-1},\gamma),$$

where $\mathcal{J}_\tau$ is the double coset

$$\mathcal{J}_\tau=\hat{\Gamma}_{\tau^{-1}\infty}\Big\backslash\hat{\Gamma}\backslash\tau^{-1}\{U^s|s\in\mathbb{Z}\}\Big/\hat{\Gamma}_\infty.$$

Remark 3.5. If $\delta_\tau\neq 0$, then $n_I=n_{\tau^{-1}}$ and $\kappa_I=\kappa_{\tau^{-1}}$.

Proof. See [Ran77] theorem 5.3.2 page 162. □

Corollary 3.15. *Let $\{f_j\}$ be an orthonormal basis of $S(\Gamma,k,\nu)$, $\tau\in\mathrm{SL}_2(\mathbb{Z})\backslash\{-U^l|l\in\mathbb{Z}\}$ and $m+\kappa_{\tau^{-1}}\geq 0$, then we have:*

$$\sum_j|(\widehat{f_j|_k\tau^{-1}})(m)|^2=\frac{\mu(\Gamma)(4\pi(m+\kappa_{\tau^{-1}}))^{k-1}}{n_{\tau^{-1}}^k\Gamma(k-1)}\times\left(1+2\pi i^{-k}\sum_{c=1}^{\infty}\frac{W(\Gamma^{\tau^{-1}},\nu^{\tau^{-1}};m,m;c)}{n_{\tau^{-1}}c}J_{k-1}\left(\frac{4\pi(m+\kappa_{\tau^{-1}})}{cn_{\tau^{-1}}}\right)\right).$$

Proof. We have

$$G_\tau(\Gamma,k,\nu;z,m) = \sum_j \langle G_\tau(\Gamma,k,\nu;\cdot,m), f_j\rangle f_j(z)$$
$$= \frac{n_{\tau^{-1}}^k \Gamma(k-1)}{\mu(\Gamma)(4\pi(m+\kappa_{\tau^{-1}}))^{k-1}} \sum_j \overline{\widehat{(f_j|_k\tau^{-1})}(m)} f_j(z)$$

and henceforth

$$\sum_j \overline{\widehat{(f_j|_k\tau^{-1})}(m)} (f_j|_k\tau^{-1})(z) = \frac{\mu(\Gamma)(4\pi(m+\kappa_{\tau^{-1}}))^{k-1}}{n_{\tau^{-1}}^k\Gamma(k-1)} (G_\tau(\Gamma,k,\nu;\cdot,m)|_k\tau^{-1})(z)$$
$$= \frac{\mu(\Gamma)(4\pi(m+\kappa_{\tau^{-1}}))^{k-1}}{n_{\tau^{-1}}^k\Gamma(k-1)} G_I(\Gamma^{\tau^{-1}},k,\nu^{\tau^{-1}};z,m).$$

Using the previous theorem we can easily deduce the m-th Fourier coefficient at ∞. To verify the equality one easily checks that $\delta_I = 1$ and n_I for $\Gamma^{\tau^{-1}}$ is equal to $n_{\tau^{-1}}$ for Γ. And by (1) also that κ_I for $\Gamma^{\tau^{-1}}, \nu^{\tau^{-1}}$ is equal to $\kappa_{\tau^{-1}}$ for Γ, ν. □

3.3 Bergman kernel

In this section we check that the construction of a reproducing kernel (Bergman kernel) for Siegel modular forms given in [Kli90] can be adjusted to give a reproducing kernel for modular forms of real weight and automorphy factor ν on Γ, a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$.
We define the Bergman kernel for $\Gamma, \nu, k > 2$ on $\mathbb{H}^2$ as:

$$h(z,w) = \sum_{\gamma\in\Gamma} \frac{1}{\left(\frac{w+\gamma z}{2i}\right)^k \nu(\gamma,z)}.$$

Theorem 3.16. *The Bergman kernel satisfies the following properties:*

1. *The sum converges absolutely uniformly on the sets* $\{z\in\mathbb{H}|\epsilon < \arg(z) < \pi-\epsilon\}\times\{w\in\mathbb{H}|\operatorname{Im} w > \epsilon\}$,
2. $\forall w\in\mathbb{H} : h(\cdot,w)\in S(\Gamma,k,\nu)$,
3. $\forall f\in S(\Gamma,k,\nu)$:
$$\langle f, h(\cdot,\overline{-w})\rangle = \frac{2}{\mu(\Gamma)}\cdot\frac{4\pi}{k-1}\cdot f(w).$$

Proof. We have for $\tau \in \mathrm{SL}_2(\mathbb{Z})$:

$$\begin{aligned}
|(h(\cdot,w)|_k\tau)(z)| &= \left| \frac{1}{j(\tau,z)^k} \sum_{\gamma\in\Gamma} \frac{1}{\left(\frac{w+\gamma\tau z}{2i}\right)^k \nu(\gamma,\tau z)} \right| \\
&\leq \sum_{\gamma\in\Gamma} \frac{1}{\left|\frac{w+\gamma\tau z}{2i}\right|^k |j(\gamma,\tau z)|^k |j(\tau,z)|^k} \\
&= \sum_{\gamma\in\Gamma} \frac{1}{\left|\frac{w+\gamma\tau z}{2i}\right|^k |j(\gamma\tau,z)|^k} \\
&\leq \sum_{\gamma\in\mathrm{SL}_2(\mathbb{Z})} \left|\frac{\operatorname{Im} w}{2}\right|^{-k} \frac{1}{|j(\gamma,z)|^k}.
\end{aligned}$$

The latter converges uniformly on the mentioned sets. It follows, that $h(\cdot,w)$ is a holomorphic function on the upper-half plane. Moreover we can exchange limit and summation in the following:

$$\begin{aligned}
\lim_{\operatorname{Im} z\to\infty} |(h(\cdot,w)|_k\tau)(z)| &\leq \lim_{\operatorname{Im} z\to\infty} \sum_{\gamma\in\mathrm{SL}_2(\mathbb{Z})} \frac{1}{\left|\frac{w+\gamma z}{2i}\right|^k |j(\gamma,z)|^k} \\
&= \sum_{\gamma\in\mathrm{SL}_2(\mathbb{Z})} \lim_{\operatorname{Im} z\to\infty} \frac{1}{\left|\frac{w+\gamma z}{2i}\right|^k |j(\gamma,z)|^k} = 0.
\end{aligned}$$

To see the latter we distinguish two cases. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If $c = 0$ then

$$\lim_{\operatorname{Im} z\to\infty} |w+\gamma z| = \lim_{\operatorname{Im} z\to\infty} |w + d^{-1}(az+b)| = \infty$$

and $j(\gamma,z) = d$. If $c \neq 0$, then $|w+\gamma z| \geq \operatorname{Im} w$ and

$$\lim_{\operatorname{Im} z\to\infty} |j(\gamma,z)| = \lim_{\operatorname{Im} z\to\infty} |cz+d| = \infty.$$

We also have for $\tau \in \Gamma$:

$$h(\tau z,w) = \sum_{\gamma\in\Gamma} \frac{1}{\left(\frac{w+\gamma\tau z}{2i}\right)^k \nu(\gamma,\tau z)} = \sum_{\gamma\in\Gamma} \frac{\nu(\tau,z)}{\left(\frac{w+\gamma\tau z}{2i}\right)^k \nu(\gamma\tau,z)} = \nu(\tau,z)h(z,w).$$

From which the second claim follows. The third claim needs more work. Let $f \in S(\Gamma,k,\nu)$. We have:

$$\begin{aligned}
f(z)\overline{h(z,\overline{-w})}y^k &= \sum_{\gamma\in\Gamma} \frac{f(z)y^k}{\left(\frac{w-\overline{\gamma z}}{2i}\right)^k \overline{\nu(\gamma,z)}} = \sum_{\gamma\in\Gamma} \frac{f(\gamma z)y^k}{\left(\frac{w-\overline{\gamma z}}{2i}\right)^k \overline{\nu(\gamma,z)}\nu(\gamma,z)} \\
&= \sum_{\gamma\in\Gamma} \frac{f(\gamma z)(\operatorname{Im}\gamma z)^k}{\left(\frac{w-\overline{\gamma z}}{2i}\right)^k}.
\end{aligned}$$

Plugging this in the definition of the Petersson inner product we find:

$$\begin{aligned}\langle f, h(\cdot, \overline{-w})\rangle &= \frac{1}{\mu(\Gamma)}\int_{\mathbb{F}_\Gamma}\sum_{\gamma\in\Gamma}\frac{f(\gamma z)(\operatorname{Im}\gamma z)^k}{\left(\frac{w-\overline{\gamma z}}{2i}\right)^k}\frac{dxdy}{y^2}\\ &= \frac{1}{\mu(\Gamma)}\sum_{\gamma\in\Gamma}\int_{\gamma\mathbb{F}_\Gamma}\frac{f(z)y^k}{\left(\frac{w-\overline{z}}{2i}\right)^k}\frac{dxdy}{y^2}\\ &= \frac{2}{\mu(\Gamma)}\int_{\mathbb{H}}\frac{f(z)y^k}{\left(\frac{w-\overline{z}}{2i}\right)^k}\frac{dxdy}{y^2}.\end{aligned}$$

Using a Cayley transformation $l_w : \mathbb{H} \to \mathbb{D}, z \mapsto \zeta = (z-w)/(z-\overline{w})$, which maps the upper-half plane biholomorphic to the unit disk, we will transform the integral. For this we denote $z = x+iy, w = u+iv, \zeta = \xi + i\eta$. We have the following identities:

$$\frac{dl_w}{dz} = \frac{w-\overline{w}}{(z-\overline{w})^2} = \frac{2v}{(z-\overline{w})^2} \Rightarrow d\xi d\eta = \frac{4v^2}{|z-\overline{w}|^4}dxdy,$$

$$1-|\zeta|^2 = \frac{|z-\overline{w}|^2 - |z-w|^2}{|z-\overline{w}|^2} = \frac{4yv}{|z-\overline{w}|^2}.$$

Back to the integral we have to calculate:

$$\begin{aligned}\int_{\mathbb{H}}\frac{f(z)y^k}{\left(\frac{w-\overline{z}}{2i}\right)^k}\frac{dxdy}{y^2} &= \int_{\mathbb{H}} f(z)\left(\frac{w-\overline{z}}{2i}\right)^{-k}\left[\frac{(1-|\zeta|^2)|z-\overline{w}|^2}{4v}\right]^k\left[\frac{4v}{(1-|\zeta|^2)|z-\overline{w}|^2}\right]^2 dxdy\\ &= \int_{\mathbb{D}} f(z)\left(\frac{w-\overline{z}}{2i}\right)^{-k}(1-|\zeta|^2)^k\left|\frac{w-\overline{z}}{2i}\right|^{2k} v^{-k}\frac{4d\xi d\eta}{(1-|\zeta|^2)^2}\\ &= \frac{4}{v^k}\int_{\mathbb{D}} f(z)\left(\frac{z-\overline{w}}{2i}\right)^k(1-|\zeta|^2)^k\frac{d\xi d\eta}{(1-|\zeta|^2)^2}\\ &= \frac{4}{v^k}\int_{\mathbb{D}} f^\dagger(\zeta)(1-|\zeta|^2)^k\frac{d\xi d\eta}{(1-|\zeta|^2)^2}.\end{aligned}$$

Where

$$f^\dagger(\zeta) = f(z)\left(\frac{z-\overline{w}}{2i}\right)^k$$

is holomorphic on $\mathbb{D}$ and satisfies

$$\left|f^\dagger(\zeta)(1-|\zeta|^2)^{\frac{k}{2}}\right| = v^{\frac{k}{2}}y^{\frac{k}{2}}|f(z)| \ll_{f,w,k} 1.$$

By computing the following integral for $0 \le t < 1, \alpha > -1$

$$\int_{t\mathbb{D}}(1-|\zeta|^2)^\alpha d\xi d\eta = \int_0^t\int_0^{2\pi}(1-r^2)^\alpha r d\varphi dr = -\pi\frac{(1-r^2)^{\alpha+1}}{\alpha+1}\bigg|_{r=0}^t = \frac{\pi}{\alpha+1}\left(1-(1-t^2)^{\alpha+1}\right)$$

we see that the integral left to be computed converges absolutely uniformly for $k > 2$. Hence we have:

$$\begin{aligned}\int_{\mathbb{D}} f^\dagger(\zeta)(1-|\zeta|^2)^k \frac{d\xi d\eta}{(1-|\zeta|^2)^2} &= \lim_{t\to 1^-} \int_{t\mathbb{D}} f^\dagger(\zeta)(1-|\zeta|^2)^k \frac{d\xi d\eta}{(1-|\zeta|^2)^2} \\ &= \lim_{t\to 1^-} \int_{t\mathbb{D}} \sum_{n=0}^{\infty} \left(f^\dagger\right)^{(n)}(0)\frac{\zeta^n}{n!}(1-|\zeta|^2)^k \frac{d\xi d\eta}{(1-|\zeta|^2)^2} \\ &= \lim_{t\to 1^-} \sum_{n=0}^{\infty} \frac{\left(f^\dagger\right)^{(n)}(0)}{n!} \int_{t\mathbb{D}} \zeta^n (1-|\zeta|^2)^k \frac{d\xi d\eta}{(1-|\zeta|^2)^2}.\end{aligned}$$

As the Taylor expansion converges absolutely uniformly on $t\mathbb{D}$. Making the substitution $\zeta \mapsto e^{2\pi i s}\zeta$ for some suitable $s \in \mathbb{R}$ we see that:

$$\int_{t\mathbb{D}} \zeta^n (1-|\zeta|^2)^k \frac{d\xi d\eta}{(1-|\zeta|^2)^2} = \begin{cases} \frac{\pi}{k-1}\left(1-(1-t^2)^{k-1}\right), & \text{if } n = 0, \\ 0, & \text{if } n > 0. \end{cases}$$

And therefore we have:

$$\int_{\mathbb{D}} f^\dagger(\zeta)(1-|\zeta|^2)^k \frac{d\xi d\eta}{(1-|\zeta|^2)^2} = f^\dagger(0)\cdot\frac{\pi}{k-1} = \frac{\pi}{k-1} f(w) v^k,$$

which completes the proof. ☐

Corollary 3.17. *Let $\{f_j\}$ be an orthonormal basis of $S(\Gamma, k, \nu)$ and $\tau \in \mathrm{SL}_2(\mathbb{Z})$, then we have:*

$$\sum_j |(f_j|_k\tau)(z)|^2 = \mu(\Gamma)\frac{k-1}{8\pi} \sum_{\gamma\in\Gamma^\tau} \frac{1}{\left(\frac{\gamma z - \bar{z}}{2i}\right)^k \nu^\tau(\gamma, z)}.$$

Proof. We prove first the case $\tau = I$, which follows easily from the formula:

$$h(z, \overline{-w}) = \sum_j \langle h(\cdot, \overline{-w}, f_j\rangle f_j(z) = \frac{2}{\mu(\Gamma)} \cdot \frac{4\pi}{k-1} \sum_j \overline{f_j(w)} f_j(z).$$

For the general case we use the special case in combination with Theorem 3.3, which shows, that $\{f_j|_k\tau\}$ is a basis of $S(\Gamma^\tau, k, \nu^\tau)$. The orthonormality follows from:

$$\begin{aligned}\langle f, g\rangle_\Gamma &= \frac{1}{\mu(\Gamma)} \int_{\mathbb{F}_\Gamma} f(z)\overline{g(z)} y^k \frac{dxdy}{y^2} \\ &= \frac{1}{\mu(\Gamma)} \int_{\tau^{-1}\mathbb{F}_\Gamma} (f|_k\tau)(z)\overline{(g|_k\tau)(z)} y^k \frac{dxdy}{y^2} \\ &= \langle f|_k\tau, g|_k\tau\rangle_{\Gamma^\tau},\end{aligned}$$

where we used $\mu(\Gamma) = \mu(\Gamma^\tau)$, the $\mathrm{SL}_2(\mathbb{Z})$ invariance of the measure $y^{-2}dxdy$ and that $\tau^{-1}\mathbb{F}_\Gamma$ is a fundamental domain for Γ^τ. ☐

3.4 Convexity bounds

In this section we give various bounds for $y^{\frac{k}{2}}|f(z)|$ for $f \in S(\Gamma, k, \nu)$ in different regions. This will require uniform results on Bessel functions, which one can find in the appendix A.

The basic idea is the following: Let $f = f_1 \in S(\Gamma, k, \nu)$ be normalized with respect to the Petersson inner product and extend it to an orthonormal basis $\{f_j\}$ of $S(\Gamma, k, \nu)$. Then we have:

$$y^{\frac{k}{2}}|f(z)| \leq \sqrt{y^k \sum_j |f_j(z)|^2}.$$

Now $y^{\frac{k}{2}}|f(z)|$ is Γ invariant, so it suffices to look for the maximum in a fundamental domain. We also have $\mathrm{Im}(\tau z)^{\frac{k}{2}}|f(\tau z)| = y^{\frac{k}{2}}|(f|_k\tau)(z)|$. Hence we have:

$$\sup_{z\in\mathbb{H}} y^{\frac{k}{2}}|f(z)| \leq \max_{\tau\in\Gamma \setminus \mathrm{SL}_2(\mathbb{Z})} \sup_{z\in\mathbb{F}_I} \sqrt{y^k \sum_j |(f_j|_k\tau)(z)|^2}, \tag{2}$$

where $\mathbb{F}_I$ is the standard fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$.

Our first method will use the Fourier expansion and a nice application of the Cauchy-Schwarz inequality to involve the Fourier coefficients of the Poincaré series:

$$\begin{aligned}
|(f_j|_k\tau)(z)|^2 &= \left| \sum_{m+\kappa_\tau>0} \widehat{(f_j|_k\tau)}(m) e^{\frac{2\pi i(m+\kappa_\tau)z}{n_\tau}} \right|^2 \\
&\leq \left(\sum_{m+\kappa_\tau>0} |\widehat{(f_j|_k\tau)}(m)| e^{-\frac{2\pi(m+\kappa_\tau)y}{n_\tau}} \right)^2 \\
&\leq \left(\sum_{m+\kappa_\tau>0} \lambda_m^{-1} |\widehat{(f_j|_k\tau)}(m)|^2 e^{-\frac{2\pi(m+\kappa_\tau)y}{n_\tau}} \right) \left(\sum_{m+\kappa_\tau>0} \lambda_m e^{-\frac{2\pi(m+\kappa_\tau)y}{n_\tau}} \right),
\end{aligned}$$

where the λ_m are positive reals to be chosen later. Summing over j we get:

$$y^k \sum_j |(f_j|_k\tau)(z)|^2 \leq \left(\sum_{m+\kappa_\tau>0} \lambda_m^{-1} A(m) y^{\frac{k}{2}} e^{-\frac{2\pi(m+\kappa_\tau)y}{n_\tau}} \right) \left(\sum_{m+\kappa_\tau>0} \lambda_m y^{\frac{k}{2}} e^{-\frac{2\pi(m+\kappa_\tau)y}{n_\tau}} \right), \tag{3}$$

where

$$A(m) = \frac{\mu(\Gamma)(4\pi(m+\kappa_\tau))^{k-1}}{n_\tau^k \Gamma(k-1)} \left(1 + 2\pi i^{-k} \sum_{c=1}^{\infty} \frac{W(\Gamma^\tau, \nu^\tau; m, m; c)}{n_\tau c} J_{k-1}\left(\frac{4\pi(m+\kappa_\tau)}{cn_\tau} \right) \right). \tag{4}$$

For the generalized Kloosterman sums we are going to use the trivial estimate:

$$|W(\Gamma^\tau, \nu^\tau; m, m, c)| \leq n_\tau^2 c.$$

Before we go any further we shall remark here that we can assume $k \gg 1$ as we want to investigate the behavior as $k \to \infty$.

We now have to deal with the sum

$$\sum_{c=1}^{\infty} \left| J_{k-1}\left(\frac{4\pi(m+\kappa_\tau)}{cn_\tau}\right)\right|.$$

We split this into the different regions:

1. $\sqrt{\frac{k-1}{2}} \geq \frac{4\pi(m+\kappa_\tau)}{cn_\tau}$,
2. $k-1-(k-1)^\alpha \geq \frac{4\pi(m+\kappa_\tau)}{cn_\tau} \geq \sqrt{\frac{k-1}{2}}$,
3. $k-1+(k-1)^\alpha \geq \frac{4\pi(m+\kappa_\tau)}{cn_\tau} \geq k-1-(k-1)^\alpha$,
4. $\frac{4\pi(m+\kappa_\tau)}{cn_\tau} \geq k-1+(k-1)^\alpha$.

Where $1 \geq \alpha > \frac{1}{3}$ yet to be chosen. For the first region we have by means of Proposition A.5:

$$\begin{aligned}
\sum_{c \geq 4\pi\sqrt{\frac{2}{k-1}}\left(\frac{m+\kappa_\tau}{n_\tau}\right)} \left| J_{k-1}\left(\frac{4\pi(m+\kappa_\tau)}{cn_\tau}\right)\right| &\leq \frac{1}{\Gamma(k)} \sum_{c \geq 4\pi\sqrt{\frac{2}{k-1}}\left(\frac{m+\kappa_\tau}{n_\tau}\right)} \left(\frac{2\pi(m+\kappa_\tau)}{cn_\tau}\right)^{k-1} \\
&\leq \frac{1}{\Gamma(k)}\left(\frac{2\pi(m+\kappa_\tau)}{n_\tau}\right)^{k-1}\left(\frac{\sqrt{2}\cdot 4\pi(m+\kappa_\tau)}{\sqrt{k-1}n_\tau}\right)^{k-1} \\
&\quad \times \left(1+\frac{\sqrt{2}\cdot 4\pi(m+\kappa_\tau)}{\sqrt{k-1}(k-2)n_\tau}\right) \\
&\ll \frac{1}{\Gamma(k)}\left(\frac{k-1}{8}\right)^{\frac{k-1}{2}}\left(1+\frac{m+\kappa_\tau}{n_\tau}\cdot k^{-\frac{3}{2}}\right) \\
&\ll k^{-\frac{k}{2}}\left(1+\frac{m+\kappa_\tau}{n_\tau}\cdot k^{-\frac{3}{2}}\right).
\end{aligned} \tag{5}$$

In the second region we have by Proposition A.6:

$$\begin{aligned}
\sum_{\substack{4\pi\sqrt{\frac{2}{k-1}}\left(\frac{m+\kappa_\tau}{n_\tau}\right)\geq c,\\ c\geq 4\pi\left(\frac{m+\kappa_\tau}{n_\tau}\right)(k-1-(k-1)^\alpha)^{-1}}} \left| J_{k-1}\left(\frac{4\pi(m+\kappa_\tau)}{cn_\tau}\right)\right| &\ll \left(\frac{m+\kappa_\tau}{n_\tau}\right) k^{-\frac{1}{2}}\cdot k^{-\frac{4}{3}} \\
&\ll \left(\frac{m+\kappa_\tau}{n_\tau}\right) k^{-\frac{11}{6}}.
\end{aligned} \tag{6}$$

For the third region we have using Proposition A.4:

$$\sum_{\substack{4\pi\left(\frac{m+\kappa_\tau}{n_\tau}\right)(k-1-(k-1)^\alpha)^{-1}\geq c,\\ c\geq 4\pi\left(\frac{m+\kappa_\tau}{n_\tau}\right)(k-1+(k-1)^\alpha)^{-1}}} \left| J_{k-1}\left(\frac{4\pi(m+\kappa_\tau)}{cn_\tau}\right)\right| \ll \left(\frac{m+\kappa_\tau}{n_\tau}\right) k^{\alpha-2}\cdot k^{-\frac{1}{3}} \ll \left(\frac{m+\kappa_\tau}{n_\tau}\right) k^{\alpha-\frac{7}{3}}. \tag{7}$$

And in the last region we have by Proposition A.8:

$$\sum_{4\pi\left(\frac{m+\kappa_\tau}{n_\tau}\right)(k-1+(k-1)^\alpha)^{-1}\geq c} \left| J_{k-1}\left(\frac{4\pi(m+\kappa_\tau)}{cn_\tau}\right)\right| \ll \left(\frac{m+\kappa_\tau}{n_\tau}\right) k^{-1}\cdot k^{-\frac{\alpha+1}{4}} \ll \left(\frac{m+\kappa_\tau}{n_\tau}\right) k^{-\frac{\alpha+5}{4}}. \tag{8}$$

We make the choice $\alpha = \frac{13}{15}$ and get for $A(m)$ (defined by equation (4)) the estimation:

$$|A(m)| \ll \frac{\mu(\Gamma)(4\pi)^k}{n_\tau^k\Gamma(k-1)}\left((m+\kappa_\tau)^{k-1}(1+n_\tau k^{-\frac{k}{2}}) + (m+\kappa_\tau)^k k^{-\frac{22}{15}}\right). \tag{9}$$

Considering the inequality (3) and the Cauchy-Schwartz equality case we should choose $\lambda_m \approx (m+\kappa_\tau)^{\frac{k}{2}}$. So lets put $\lambda_m = (m+\kappa_\tau)^{\frac{k}{2}+\delta}$ with $\delta = o(k)$. The sum

$$S(\alpha,\beta,\kappa) = \sum_{m+\kappa>0}(m+\kappa)^\alpha e^{-\beta(m+\kappa)}, \quad \alpha,\beta,\kappa>0 \tag{10}$$

appears often in the next few calculations, hence the following lemma will be useful.

Lemma 3.18. *$S(\alpha,\beta,\kappa)$ as defined by* (10) *satisfies the following inequalities:*

$$S(\alpha,\beta,\kappa) \leq \beta^{-\alpha-1}\Gamma(\alpha+1) + \beta^{-\alpha}\alpha^\alpha e^{-\alpha}$$

and for $\alpha \leq \beta\kappa$ we have:

$$S(\alpha,\beta,\kappa) \leq \beta^{-\alpha-1}\Gamma(\alpha+1) + \kappa^\alpha e^{-\beta\kappa}.$$

Proof. The function $x^\alpha e^{-\beta x}$ increases on $(0,\frac{\alpha}{\beta}]$ and decreases on $[\frac{\alpha}{\beta},\infty)$. Hence we get

$$\begin{aligned} S(\alpha,\beta,\kappa) &\leq \int_\kappa^\infty x^\alpha e^{-\beta x}dx + \left(\frac{\alpha}{\beta}\right)^\alpha e^{-\beta\frac{\alpha}{\beta}} \\ &\leq \int_0^\infty x^\alpha e^{-\beta x}dx + \beta^{-\alpha}\alpha^\alpha e^{-\alpha} \\ &= \beta^{-\alpha-1}\Gamma(\alpha+1) + \beta^{-\alpha}\alpha^\alpha e^{-\alpha}. \end{aligned}$$

And if one assumes $\alpha \le \beta\kappa$, then:

$$\begin{aligned} S(\alpha,\beta,\kappa) &\le \int_\kappa^\infty x^\alpha e^{-\beta x}dx + \kappa^\alpha e^{-\beta\kappa} \\ &\le \beta^{-\alpha-1}\Gamma(\alpha+1) + \kappa^\alpha e^{-\beta\kappa}. \end{aligned}$$

□

Using (9) in (3) with the choice $\lambda_m = (m+\kappa_\tau)^{\frac{k}{2}+\delta}$ we get:

$$\begin{aligned} y^k \sum_j |(f_j|_k\tau)(z)|^2 \le{}& \frac{y^k(4\pi)^k\mu(\Gamma)}{n_\tau^k\Gamma(k-1)} S\left(\frac{k}{2}+\delta, \frac{2\pi y}{n_\tau}, \kappa_\tau\right) \\ &\times \left((1+n_\tau k^{-\frac{k}{2}})S\left(\frac{k}{2}-\delta-1, \frac{2\pi y}{n_\tau}, \kappa_\tau\right) + k^{-\frac{22}{15}} S\left(\frac{k}{2}-\delta, \frac{2\pi y}{n_\tau}, \kappa_\tau\right)\right). \end{aligned} \tag{11}$$

Using Lemma 3.18 we have:

$$\begin{aligned} S\left(\frac{k}{2}+\delta, \frac{2\pi y}{n_\tau}, \kappa_\tau\right) &\le \left(\frac{2\pi y}{n_\tau}\right)^{-\frac{k}{2}-\delta-1} \Gamma\left(\frac{k}{2}+\delta+1\right) + \left(\frac{2\pi y}{n_\tau}\right)^{-\frac{k}{2}-\delta} \left(\frac{k}{2}+\delta\right)^{\frac{k}{2}+\delta} e^{-\frac{k}{2}-\delta} \\ &\ll \left(\frac{2\pi y}{n_\tau}\right)^{-\frac{k}{2}-\delta-1} \left(\frac{k}{2}\right)^{\frac{k+1}{2}+\delta} e^{-\frac{k}{2}-\delta} \left[e^\delta + \frac{2\pi y}{n_\tau} k^{-\frac{1}{2}} e^\delta\right] \\ &\ll \frac{(4\pi)^{-\frac{k}{2}-\delta} y^{-\frac{k}{2}-\delta-1} k^{\frac{k+1}{2}+\delta} e^{-\frac{k}{2}}}{n_\tau^{-\frac{k}{2}-\delta-1}} \left[1 + \frac{yk^{-\frac{1}{2}}}{n_\tau}\right], \\ S\left(\frac{k}{2}-\delta-1, \frac{2\pi y}{n_\tau}, \kappa_\tau\right) &\ll \frac{(4\pi)^{-\frac{k}{2}+\delta} y^{-\frac{k}{2}+\delta} k^{\frac{k-1}{2}-\delta} e^{-\frac{k}{2}}}{n_\tau^{-\frac{k}{2}+\delta}} \left[1 + \frac{yk^{-\frac{1}{2}}}{n_\tau}\right], \\ S\left(\frac{k}{2}-\delta, \frac{2\pi y}{n_\tau}, \kappa_\tau\right) &\ll \frac{(4\pi)^{-\frac{k}{2}+\delta} y^{-\frac{k}{2}+\delta-1} k^{\frac{k+1}{2}-\delta} e^{-\frac{k}{2}}}{n_\tau^{-\frac{k}{2}+\delta-1}} \left[1 + \frac{yk^{-\frac{1}{2}}}{n_\tau}\right]. \end{aligned}$$

Plugging these inequalities into (11) we get:

Proposition 3.19. *Let ν be an automorphy factor of weight $k \gg 1$ for Γ a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$, $\tau \in \mathrm{SL}_2(\mathbb{Z})$ and $\{f_j\}$ an orthonormal basis of $S(\Gamma,k,\nu)$ then we have for $z \in \mathbb{F}_I$:*

$$y^k \sum_j |(f_j|_k\tau)(z)|^2 \ll \frac{\mu(\Gamma) n_\tau k^{\frac{3}{2}}}{y} \left[1 + \frac{yk^{-\frac{1}{2}}}{n_\tau}\right]^2 \left[1 + n_\tau k^{-\frac{k}{2}} + \frac{n_\tau}{y} k^{-\frac{7}{15}}\right]. \tag{12}$$

For large y we can improve on this. For this purpose we assume $|\delta| + 1 \le \frac{k}{2}$ and $y \ge \frac{3n_\tau k}{\kappa_\tau \pi}$. These assumptions will allow us to use the following lemma.

Lemma 3.20. *The following inequality holds for $x \geq 6\frac{\alpha}{\beta},\ \alpha, \beta > 0$:*

$$x^\alpha e^{-\beta x} \leq \alpha^\alpha \beta^{-\alpha} e^{-\alpha} \cdot e^{-\frac{\beta x}{2}}.$$

Proof. Let $x = c\frac{\alpha}{\beta}$, then

$$x^\alpha e^{-\beta x} = \alpha^\alpha \beta^{-\alpha} e^{-\alpha} \cdot e^{-\frac{\beta x}{2}} \cdot \left(c e^{1-\frac{c}{2}}\right)^\alpha.$$

Note that $6e^{1-3} < 1$ and that $ce^{1-\frac{c}{2}}$ is decreasing on $[2, \infty)$. $\square$

Using Lemmata 3.18 and 3.20 we get:

$$\begin{aligned}
S\left(\frac{k}{2}+\delta, \frac{2\pi y}{n_\tau}, \kappa_\tau\right) &\leq \left(\frac{2\pi y}{n_\tau}\right)^{-\frac{k}{2}-\delta-1} \Gamma\left(\frac{k}{2}+\delta+1\right) + \kappa_\tau^{\frac{k}{2}+\delta} e^{-\frac{2\pi y}{n_\tau}\kappa_\tau} \\
&\ll \left(\frac{2\pi y}{n_\tau}\right)^{-\frac{k}{2}-\delta-1} \left(\frac{k}{2}\right)^{\frac{k+1}{2}+\delta} e^{-\frac{k}{2}} \\
&\quad \times \left[1 + \left(\frac{2\pi y}{n_\tau}\right)^{\frac{k}{2}+\delta+1} \left(\frac{k}{2}\right)^{-\frac{k+1}{2}-\delta} e^{\frac{k}{2}} \kappa_\tau^{\frac{k}{2}+\delta} e^{-\frac{2\pi y}{n_\tau}\kappa_\tau}\right] \\
&\ll \left(\frac{2\pi y}{n_\tau}\right)^{-\frac{k}{2}-\delta-1} \left(\frac{k}{2}\right)^{\frac{k+1}{2}+\delta} e^{-\frac{k}{2}} \left[1 + \kappa_\tau^{-1} k^{\frac{1}{2}} e^{-\frac{\kappa_\tau \pi}{n_\tau} y}\right], \\
S\left(\frac{k}{2}-\delta-1, \frac{2\pi y}{n_\tau}, \kappa_\tau\right) &\ll \left(\frac{2\pi y}{n_\tau}\right)^{-\frac{k}{2}+\delta} \left(\frac{k}{2}\right)^{\frac{k+1}{2}-\delta-1} e^{-\frac{k}{2}} \left[1 + \kappa_\tau^{-1} k^{\frac{1}{2}} e^{-\frac{\kappa_\tau \pi}{n_\tau} y}\right], \\
S\left(\frac{k}{2}-\delta, \frac{2\pi y}{n_\tau}, \kappa_\tau\right) &\ll \left(\frac{2\pi y}{n_\tau}\right)^{-\frac{k}{2}+\delta-1} \left(\frac{k}{2}\right)^{\frac{k+1}{2}-\delta} e^{-\frac{k}{2}} \left[1 + \kappa_\tau^{-1} k^{\frac{1}{2}} e^{-\frac{\kappa_\tau \pi}{n_\tau} y}\right].
\end{aligned}$$

Plugging these inequalities into (11) we get:

Proposition 3.21. *Let ν be an automorphy factor of weight $k \gg 1$ for Γ a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$, $\tau \in \mathrm{SL}_2(\mathbb{Z})$ and $\{f_j\}$ an orthonormal basis of $S(\Gamma, k, \nu)$ then we have for $z \in \mathbb{F}_I, y \geq \frac{3n_\tau k}{\kappa_\tau \pi}$:*

$$y^k \sum_j |(f_j|_k\tau)(z)|^2 \ll \frac{\mu(\Gamma) n_\tau k^{\frac{3}{2}}}{y} \left[1 + \kappa_\tau^{-1} k^{\frac{1}{2}} e^{-\frac{\kappa_\tau \pi}{n_\tau} y}\right]^2 \left[1 + n_\tau k^{-\frac{k}{2}} + \frac{n_\tau}{y} k^{-\frac{7}{15}}\right]. \tag{13}$$

Our second method will be based on the Bergman kernel, in particular the identity of the Corollary 3.17. For this purpose it is sufficient to bound the following sum for $z \in \mathbb{F}_I$:

$$\sum_{\gamma \in \Gamma^\tau} \frac{y^k}{\left(\frac{\gamma z - \bar{z}}{2i}\right)^k \nu^\tau(\gamma, z)} \leq \sum_{\gamma \in \mathrm{SL}_2(\mathbb{Z})} \frac{y^k}{\left|\frac{\gamma z - \bar{z}}{2i}\right|^k |j(\gamma, z)|^k} = \sum_{\gamma \in \mathrm{SL}_2(\mathbb{Z})} \frac{(yy')^{\frac{k}{2}}}{\left(\left(\frac{x-x'}{2}\right)^2 + \left(\frac{y+y'}{2}\right)^2\right)^{\frac{k}{2}}}, \tag{14}$$

where $x' + iy' = z' = \gamma z$. Our first estimation will be crude and will be used to deal with the cases $y \ll 1$ and $k = 3$, which then allows a good treatment of the sum, when y is relatively small in comparison to k. First notice, that by AM-GM we have

$$\left(\frac{y+y'}{2}\right)^2 \geq yy'.$$

Thus every term is ≤ 1. For $c > 0, (c,d) = 1$ fix a matrix $\gamma_{c,d} = \begin{pmatrix} * & * \\ c & d \end{pmatrix}$ with $|\operatorname{Re}(\gamma_{c,d}z - z)| \leq \frac{1}{2}$ then we have:

$$\sum_{\gamma \in \mathrm{SL}_2(\mathbb{Z})} \frac{(yy')^{\frac{k}{2}}}{\left(\left(\frac{x-x'}{2}\right)^2 + \left(\frac{y+y'}{2}\right)^2\right)^{\frac{k}{2}}} \leq 2+2 \sum_{\substack{c>0,\\(c,d)=1}} \sum_{b\in\mathbb{Z}} \frac{(yy'')^{\frac{k}{2}}}{\left(\left(\frac{x-x''-b}{2}\right)^2 + \left(\frac{y+y''}{2}\right)^2\right)^{\frac{k}{2}}} + 2\sum_{b>0} \frac{y^k}{\left(\left(\frac{b}{2}\right)^2 + y^2\right)^{\frac{k}{2}}}$$

where $x'' + iy'' = \gamma_{c,d}z$. Recall the assumption $k > 2$ for the Bergman kernel. For the last sum we have:

$$\begin{aligned}\sum_{b>0} \frac{y^k}{\left(\left(\frac{b}{2}\right)^2 + y^2\right)^{\frac{k}{2}}} &\leq \sum_{0<b<2y+1} 1 + \sum_{b\geq 2y+1} \frac{y^k}{\left(\frac{b}{2}\right)^k} \\ &\leq 2y + 1 + \int_{2y}^{\infty} \frac{2^k y^k}{u^k} du \\ &\ll y.\end{aligned}$$

For the inner sum of the middle sum we have:

$$\begin{aligned}\sum_{b\in\mathbb{Z}} \frac{(yy'')^{\frac{k}{2}}}{\left(\left(\frac{x-x''-b}{2}\right)^2 + \left(\frac{y+y''}{2}\right)^2\right)^{\frac{k}{2}}} &\leq \sum_{|b|<y+y''+2} \frac{(yy'')^{\frac{k}{2}}}{\left(\frac{y+y''}{2}\right)^k} + 2 \sum_{b\geq y+y''+1} \frac{(yy'')^{\frac{k}{2}}}{\left(\frac{b}{2}\right)^k} \\ &\leq (2(y+3)+1) \frac{(yy'')^{\frac{k}{2}}}{\left(\frac{y+y''}{2}\right)^k} + (yy'')^{\frac{k}{2}} \int_{y+y''}^{\infty} \frac{2^k}{u^k} du \\ &\ll y \frac{(yy'')^{\frac{k}{2}}}{\left(\frac{y+y''}{2}\right)^k}.\end{aligned}$$

Summing this over the outer sum we get:

$$\begin{aligned}\sum_{\substack{c>0,\\(c,d)=1}}\frac{(yy'')^{\frac{k}{2}}}{\left(\frac{y+y''}{2}\right)^k} &= \sum_{\substack{c>0,\\(c,d)=1}}\left(\frac{|cz+d|+\frac{1}{|cz+d|}}{2}\right)^{-k} \le \sum_{c>0}\sum_{d\in\mathbb{Z}}\left(\frac{|cz+d|+\frac{1}{|cz+d|}}{2}\right)^{-k}\\
&\le \sum_{c=1}^{3}\sum_{d\in\mathbb{Z}}\left(\frac{|cz+d|+\frac{1}{|cz+d|}}{2}\right)^{-k} + \sum_{c\ge 4}\sum_{d\in\mathbb{Z}}\left(\frac{|cz+d|+\frac{1}{|cz+d|}}{2}\right)^{-k}\\
&\le \sum_{c=1}^{3}\sum_{|d-cx|<2y+4}\left(\frac{|cz+d|+\frac{1}{|cz+d|}}{2}\right)^{-k} + 2\sum_{c=1}^{3}\sum_{d\ge 2y+3}\left(\frac{d}{2}\right)^{-k}\\
&\quad + \sum_{c\ge 4}\sum_{|d-cx|<2cy+3}\left(\frac{cy}{2}\right)^{-k} + 2\sum_{c\ge 4}\sum_{d\ge 2cy+1}\left(\frac{d}{2}\right)^{-k}\\
&\ll y + \int_{2y+2}^{\infty}\left(\frac{t}{2}\right)^{-k}dt + \int_{3}^{\infty}\left(\frac{sy}{2}\right)^{1-k}ds + \int_{3}^{\infty}\int_{2sy}\left(\frac{t}{2}\right)^{-k}dtds\\
&\ll y\left(1+\frac{1}{k-2}\right).\end{aligned}$$

So we have

$$\sum_{\gamma\in\mathrm{SL}_2(\mathbb{Z})}\frac{(yy')^{\frac{k}{2}}}{\left(\left(\frac{x-x'}{2}\right)^2+\left(\frac{y+y'}{2}\right)^2\right)^{\frac{k}{2}}} \ll y\left(1+\frac{1}{k-2}\right). \tag{15}$$

We now assume $y\ge 3, k\ge 6$, then we have:

$$\begin{aligned}\sum_{\gamma\in\mathrm{SL}_2(\mathbb{Z})}\frac{(yy')^{\frac{k}{2}}}{\left(\left(\frac{x-x'}{2}\right)^2+\left(\frac{y+y'}{2}\right)^2\right)^{\frac{k}{2}}} &\le 2+8N\frac{y^k}{\left(\frac{1}{4}+y^2\right)^{\frac{k}{2}}}\\
&\quad + \sum_{\gamma\in\mathrm{SL}_2(\mathbb{Z})\setminus\{\pm U^n|\ |n|\le 2N\}}\frac{(yy')^{\frac{k}{2}}}{\left(\left(\frac{x-x'}{2}\right)^2+\left(\frac{y+y'}{2}\right)^2\right)^{\frac{k}{2}}}\\
&\ll 1+N\frac{y^k}{\left(\frac{1}{4}+y^2\right)^{\frac{k}{2}}}\\
&\quad + y\sup_{\gamma\in\mathrm{SL}_2(\mathbb{Z})\setminus\{\pm U^n|\ |n|\le 2N\}}\frac{(yy')^{\frac{k-3}{2}}}{\left(\left(\frac{x-x'}{2}\right)^2+\left(\frac{y+y'}{2}\right)^2\right)^{\frac{k-3}{2}}}.\end{aligned} \tag{16}$$

We need to estimate this supremum. If $\gamma = \pm U^n$, then $n \geq 2N$ and we have:

$$\frac{(yy')^{\frac{k-3}{2}}}{\left(\left(\frac{x-x'}{2}\right)^2 + \left(\frac{y+y'}{2}\right)^2\right)^{\frac{k-3}{2}}} \leq \frac{y^{k-3}}{(N^2+y^2)^{\frac{k-3}{2}}} = \left(1 + \frac{N^2}{y^2}\right)^{-\frac{k-3}{2}}.$$

Otherwise we have

$$\frac{(yy')^{\frac{k-3}{2}}}{\left(\left(\frac{x-x'}{2}\right)^2 + \left(\frac{y+y'}{2}\right)^2\right)^{\frac{k-3}{2}}} \leq \frac{(yy')^{\frac{k-3}{2}}}{\left(\frac{y+y'}{2}\right)^{k-3}} = \left(\frac{|cz+d| + \frac{1}{|cz+d|}}{2}\right)^{3-k} \leq \left(\frac{y}{2}\right)^{3-k}.$$

We want both of them to have faster decay than any polynomial in k. This is because for large y we can use Proposition 3.21 to get better estimates. This fast decay is given if $N \geq \frac{y}{k^{\frac{1}{2}-\eta}}$ for some $\frac{1}{2} > \eta > 0$. We regard η as a fixed constant. We thus have:

$$\sup_{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \setminus \{\pm U^n \mid |n| \leq 2N\}} \frac{(yy')^{\frac{k-3}{2}}}{\left(\left(\frac{x-x'}{2}\right)^2 + \left(\frac{y+y'}{2}\right)^2\right)^{\frac{k-3}{2}}} \ll e^{-\delta k^{\frac{\eta}{2}}} \tag{17}$$

for some $\delta > 0$. For $\frac{\log y}{\log k} \leq \frac{1}{2} - \eta$ the factor

$$\frac{y^k}{\left(\frac{1}{4} + y^2\right)^{\frac{k}{2}}}$$

has the same decay (change δ if necessary). One verifies, that $N = \frac{y}{k^{\frac{1}{2}-\eta}}$ is about best possible. We summarize the estimations in the following proposition.

Proposition 3.22. *Let ν be an automorphy factor of weight $k \geq 6$ for a finite index subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$, $\tau \in \mathrm{SL}_2(\mathbb{Z})$ and $\{f_j\}$ an orthonormal basis of $S(\Gamma, k, \nu)$. Fix $\frac{1}{2} > \eta > 0$ then we have for $z \in \mathbb{F}_I$:*

$$y^k \sum_j |(f_j|_k\tau)(z)|^2 \ll_\eta \mu(\Gamma) k \left(1 + \frac{y}{k^{\frac{1}{2}-\eta}}\right).$$

This improves Proposition 3.19, when y is small. And gives the following theorem:

Theorem 3.23. *Let ν be an automorphy factor of weight $k \geq 6$ for a finite index subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ and $f \in S(\Gamma, k, \nu)$ with Petersson norm $\langle f, f\rangle_\Gamma = 1$. Then we have for any compact subset $K \subseteq \mathbb{H}$:*

$$\sup_{z \in K} y^{\frac{k}{2}} |f(z)| \ll_K \mu(\Gamma)^{\frac{1}{2}} k^{\frac{1}{2}}.$$

This generalizes a result of Rudnick [Rud05] to arbitrary finite index subgroups and real automorphy factors.

One should also remark here, that the estimations in the Propositions 3.19, 3.21 and 3.22 combined are rather good if one takes into account the dimension of the space. As one would expect that the following holds:

Let $m \in \mathbb{N}$ be a natural number and $\nu_1, \ldots, \nu_m$ be automorphy factors of weight $k_1, \ldots, k_m > 0$ on $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ a finite index subgroup, such that $M(\Gamma, k_i, \nu_i) \neq \{0\}$ for $i = 1, \ldots, m$. Then there exist constants $C_1, C_2 > 0$ such that for any finite index subgroup $\Gamma' \subseteq \Gamma$ and any $n_1, \ldots, n_m \in \mathbb{N}_0$ we have:

$$C_1 \mu(\Gamma') k_1^{n_1} \ldots k_m^{n_m} \leq \dim M(\Gamma', k_1^{n_1} \ldots k_m^{n_m}, \nu_1^{n_1} \ldots \nu_m^{n_m}) \leq C_2 \mu(\Gamma') k_1^{n_1} \ldots k_m^{n_m}.$$

Remark 3.6. The upper bound has been proven in [Ran77] theorem 4.2.1 page 102.

Using the three Propositions 3.19, 3.21, 3.22 with the equation (2) we deduce the following theorem.

Theorem 3.24. *Let ν be an automorphy factor of weight $k \gg 1$ with respect to a finite index subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$. Further let $f \in S(\Gamma, k, \nu)$ with $\langle f, f \rangle_\Gamma = 1$ be a normalized cusp form. Then we have:*

$$\sup_{z \in \mathbb{H}} y^{\frac{k}{2}} |f(z)| \ll_\epsilon \max_{\tau \in \Gamma \setminus \mathrm{SL}_2(\mathbb{Z})} \left\{ \max\{n_\tau^{\frac{1}{4}}, \kappa_\tau^{-\frac{1}{2}}\} \left(1 + n_\tau k^{-\frac{23}{16}+\epsilon}\right)^{\frac{1}{2}} \right\} \cdot \mu(\Gamma)^{\frac{1}{2}} k^{\frac{3}{4}},$$

$$\sup_{z \in \mathbb{H}} y^{\frac{k}{2}} |f(z)| \ll_\epsilon \left(1 + \max_{\tau \in \Gamma \setminus \mathrm{SL}_2(\mathbb{Z})} n_\tau^{\frac{1}{2}} k^{-\frac{1}{2}+\epsilon}\right) \left(\max_{\tau \in \Gamma \setminus \mathrm{SL}_2(\mathbb{Z})} \kappa_\tau^{-\frac{1}{2}}\right) \cdot \mu(\Gamma)^{\frac{1}{2}} k^{\frac{3}{4}}.$$

Proof. For $y \leq n_\tau^{\frac{1}{2}} k^{1-\epsilon}$ we use Proposition 3.22, for $y \geq \frac{3 n_\tau k}{\kappa_\tau \pi}$ we use Proposition 3.21 and for y in between we use Proposition 3.19. For the second one we use for $y \leq \max_{\tau \in \Gamma \setminus \mathrm{SL}_2(\mathbb{Z})} n_\tau$ we use Proposition 3.22, for $y \geq \frac{3 n_\tau k}{\kappa_\tau \pi}$ we use Proposition 3.21 and for y in between we use Proposition 3.19. ☐

If we reduce this theorem to a classical setting with $\Gamma = \Gamma_0(N)$ and automorphy factor χj^{2n} for $n \in \mathbb{N}$, we get the following theorem.

Theorem 3.25. *Let $k \in \mathbb{Z}, k \geq 6, N \in \mathbb{N}, \chi$ a character modulo N, such that $\chi(-1) = (-1)^k$, and $f \in S(\Gamma_0(N), k, \chi j^k)$ normalized such that $\langle f, f \rangle_{\Gamma_0(N)} = 1$. Then we have in complete uniformity:*

$$\sup_{z \in \mathbb{H}} y^{\frac{k}{2}} |f(z)| \ll_\epsilon N^{\frac{1}{2}} \log\log(16N)^{\frac{1}{2}} \left(1 + N^{\frac{1}{2}} k^{-\frac{1}{2}+\epsilon}\right) k^{\frac{3}{4}}.$$

Proof. We have $n_\tau | N$ and $\kappa_\tau = 1$ for all $\tau \in \mathrm{SL}_2(\mathbb{Z})$. And $\mu(\Gamma_0(N)) = N \prod_{p|N}(1 + p^{-1}) \ll N \log\log(16N)$. So it follows as an immediate corollary of the second inequality of the previous theorem. ☐

One may believe that the exponent $\frac{3}{4}$ is the best possible by having a look at the case $N = 1$,χ trivial and $k \in 2\mathbb{N}$. Consider the function $f(z) = \dim S(\mathrm{SL}_2(\mathbb{Z}), k, j^k)^{-\frac{1}{2}} \sum_j \mathrm{sign}(\overline{\hat{f}_j(1)}) f_j(z)$, where $\{f_j\}$ is an orthonormal basis of Hecke eigenforms. This function f satisfies $\sup_{z\in\mathbb{H}} y^{\frac{k}{2}}|f(z)| \gg_\epsilon k^{\frac{3}{4}-\epsilon}$ and $\langle f, f\rangle_{\mathrm{SL}_2(\mathbb{Z})} = 1$.

We now come to some lower bounds on average. We have:

$$\begin{aligned} y^{\frac{k}{2}} \left|\widehat{(f|_k\tau)}(m)\right| &= \frac{1}{n_\tau}\left|\int_0^{n_\tau} y^{\frac{k}{2}} (f|_k\tau)(z) e^{-\frac{2\pi i(m+\kappa_\tau)z}{n_\tau}} dx\right| \\ &\leq \frac{1}{n_\tau} e^{\frac{2\pi(m+\kappa_\tau)}{n_\tau}y} \cdot \int_0^{n_\tau} y^{\frac{k}{2}} \left|(f|_k\tau)(z)\right| dx. \end{aligned} \tag{18}$$

If we sum the squares of this inequality over an orthonormal basis $\{f_j\}$ we can use the Fourier coefficients of the Pioncaré series (see Corollary 3.15) for the left hand side and for the right hand side we can use Cauchy-Schwarz to get:

$$\begin{aligned} \sup_{\mathrm{Im}\, z=y} \sum_j y^k |(f_j|_k\tau)(z)|^2 &\geq \frac{1}{n_\tau^2}\sum_j \left(\int_0^{n_\tau} dx\right)\left(\int_0^{n_\tau} y^k|(f_j|_k\tau)(z)|^2 dx\right) \\ &\geq \frac{1}{n_\tau^2}\sum_j \left(\int_0^{n_\tau} y^{\frac{k}{2}} \left|(f|_k\tau)(z)\right| dx\right)^2 \\ &\geq y^k e^{-\frac{4\pi(m+\kappa_\tau)}{n_\tau}y} \cdot \sum_j \left|\widehat{(f|_k\tau)}(m)\right|^2 \\ &\geq y^k e^{-\frac{4\pi(m+\kappa_\tau)}{n_\tau}y} \cdot \frac{\mu(\Gamma)(4\pi(m+\kappa_\tau))^{k-1}}{n_\tau^k \Gamma(k-1)} \\ &\quad \times \left(1 + 2\pi i^{-k}\sum_{c=1}^\infty \frac{W(\Gamma^\tau, \nu^\tau; m, m; c)}{n_\tau c} J_{k-1}\left(\frac{4\pi(m+\kappa_\tau)}{cn_\tau}\right)\right). \end{aligned} \tag{19}$$

For $k \geq 320(m+1)^2$ we can use Proposition A.5 to estimate that the right hand side is bigger or equal to

$$y^k e^{-\frac{4\pi(m+\kappa_\tau)}{n_\tau}y} \cdot \frac{\mu(\Gamma)(4\pi(m+\kappa_\tau))^{k-1}}{n_\tau^k\Gamma(k-1)}\left(1 - 2\pi n_\tau \frac{\zeta(k-1)}{\Gamma(k)}\left(\frac{2\pi(m+\kappa_\tau)}{n_\tau}\right)^{k-1}\right).$$

If one takes $y = \frac{kn_\tau}{4\pi(m+\kappa_\tau)}, m = 0$ and τ such that κ_τ is minimal. We get the following theorem:

Theorem 3.26. *Let ν be an automorphy factor of weight $k \geq 320$ for Γ a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$, and $\{f_j\}$ an orthonormal basis of $S(\Gamma, k, \nu)$. Then we have:*

$$\sum_j \sup_{z\in\mathbb{H}} y^k|f_j(z)|^2 \geq \sup_{z\in\mathbb{H}}\sum_j y^k|f_j(z)|^2 \gg \frac{\mu(\Gamma)k^{\frac{3}{2}}}{\min\limits_{\tau\in\mathrm{SL}_2(\mathbb{Z})}\kappa_\tau} \times \left(1 - O\left(\left(\frac{2\pi e}{k}\right)^{k-1}\right)\right).$$

This coincides almost with the "leading" term given in Theorem 3.24. In particular we have the following corollary.

Corollary 3.27. *Let ν be an automorphy factor of weight $k \gg 1$ for Γ a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Then we have for $\mu(\Gamma) \ll k^{1-\eta}$ and $\{f_j\}$ an orthonormal basis of $S(\Gamma, k, \nu)$:*

$$\sup_{z\in\mathbb{H}} \sum_j y^k |f_j(z)|^2 \asymp_\eta \frac{\mu(\Gamma) k^{\frac{3}{2}}}{\min\limits_{\tau\in\mathrm{SL}_2(\mathbb{Z})} \kappa_\tau},$$

where the implied constant depends at most on η and the implied constant in $\mu(\Gamma) \ll k^{1-\eta}$.

Proof. This is just a combination of the Theorems 3.24 and 3.26 by noting that each cusp width is at most as big as the index. □

4 Modular forms of half-integral weight

As the name suggests in this section we restrict ourselves to modular forms of half-integral weight k, i.e. $k \in \frac{1}{2}\mathbb{Z}$. Which of course gives rise to the question: with respect to which automorphy factor? The simplest function one can think of, which has an automorphy factor of weight $\frac{1}{2}$, is the Θ-function given by:

$$\Theta(z) = \sum_{n\in\mathbb{Z}} e^{2\pi i n^2 z}.$$

It gives rise to an automorphy factor for the subgroup $\Gamma_0(4)$. Due to it's importance in number theory, it is reasonable to define the corresponding automorphy factor

$$j_\Theta(\gamma, z) = \frac{\Theta(\gamma z)}{\Theta(z)}$$

as the standard automorphy factor of weight $\frac{1}{2}$ for $\Gamma_0(4)$. Which means if we talk about a modular form of half-integral weight k we mean an element of $M(\Gamma_0(4), k, j_\Theta^{2k})$. An explicit formula of j_Θ has been given by Hecke:

Theorem 4.1 (Hecke)**.** *The automorphy factor j_Θ corresponding to the Θ-function is given by:*

$$j_\Theta(\gamma, z) = \left(\frac{c}{d}\right)\epsilon_d^{-1} j(\gamma, z)^{\frac{1}{2}}, \quad \forall\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4), \forall z \in \mathbb{H},$$

where

$$\epsilon_d = \begin{cases} 1, & \text{if } d \equiv 1 \bmod (4), \\ i, & \text{if } d \equiv 3 \bmod (4), \end{cases}$$

and $\left(\frac{c}{d}\right)$ is the Jacobi symbol if $d > 0$ and for $d < 0$ it is given by:

$$\left(\frac{c}{d}\right) = \eta \cdot \left(\frac{c}{-d}\right) \text{ with } \eta = \begin{cases} 1, & \text{if } c > 0, \\ -1, & \text{if } c < 0. \end{cases}$$

Proof. See [Kob93] page 148 and following. □

From this automorphy factor one easily constructs more automorphy factors on $\Gamma_0(4N)$ of weight k, which are given by:

$$\chi(d) j_\theta(\gamma, z)^{2k}, \quad \forall\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4N), \forall z \in \mathbb{H},$$

where χ is an even character modulo $4N$. We denote this automorphy factor by $j_{\theta,\chi}^{2k}$.

We should also remark here that contrary to the previous chapter we define the cusp parameter κ_τ to be in $[0, 1)$. This affects our definition of the Fourier coefficients in such a way that they are defined in the same way as in common literature. This has the effect that the following theorems look more natural.

4.1 Hecke operators

In this section we give an overview of the Hecke operators in the half-integral case for $\Gamma_0(4N)$ and $j^{2k}_{\Theta,\chi}$. More details and generality can be found in [Shi73] and [Kob93]. As in the integral case, Hecke operators for modular forms of half-integral weight can be defined by double cosets, but contrary to the integral case one has to work with a bigger group $\mathfrak{G}$, whose elements are of the type (γ, φ), where $\gamma \in \mathrm{GL}_2^+(\mathbb{R})$ and φ is a holomorphic function on $\mathbb{H}$ satisfying:

$$|\varphi(z)| = (\det \gamma)^{-\frac{1}{4}} |j(\gamma, z)|^{\frac{1}{2}}.$$

$\mathfrak{G}$ is then made into a group by the multiplication law

$$(\gamma, \varphi)(\gamma', \varphi') = (\gamma\gamma', (\varphi \circ \gamma') \cdot \varphi').$$

From now on we assume, that k is truely half-integral, i.e. $k \in \frac{1}{2} + \mathbb{Z}$. We can now extend our definition of the "|" action to $\mathfrak{G}$, by defining

$$(f|_k(\gamma, \varphi))(z) = \varphi^{-2k}(z) f(\gamma z).$$

We have an inclusion of $\Gamma_0(4)$ in $\mathfrak{G}$ given by $\gamma \mapsto \gamma^* = (\gamma, j_\Theta(\gamma, \cdot))$. We go on and define the m-th Hecke operator for $S(\Gamma_0(4N), k, j^{2k}_{\Theta,\chi})$ as:

$$f|_k T(m) = \sum_{\xi \in \Gamma_0(4N)^* \setminus \Gamma_0(4N)^* \xi_{1,m} \Gamma_0(4N)^*} f|_k \xi,$$

where

$$\xi_{1,m} = \left(\begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix}, m^{\frac{1}{4}} \right).$$

These operators are well defined. $T(m), T(m')$ commute for $(m, m') = 1$. They are hermitian with respect to the Petersson inner product and are thus simultaneously diagonalizable. They are the 0 map for m not a square and $T(p^2)$ has the following effect on the Fourier coefficients:

$$(\widehat{f|_k T(p^2)})(n) = \hat{f}(p^2 n) + \chi(p) \left(\frac{(-1)^{k-\frac{1}{2}} n}{p} \right) p^{k-\frac{3}{2}} \hat{f}(n) + \chi(p^2) p^{2k-2} \hat{f}(n/p^2), \tag{20}$$

where $\hat{f}(n/p^2) = 0$ if $p^2 \not| n$ and $\chi(p)\left(\frac{(-1)^{k-\frac{1}{2}} n}{p}\right) = 0$ for $p = 2$.
For a Hecke eigenform f with eigenvalues ω_p (with respect to $T(p^2)$) the above gives rise to Euler products for t square-free and relatively prime to $4N$:

$$\sum_{n=1}^{\infty} \hat{f}(tn^2) n^{-s} = \hat{f}(t) \prod_p \left[1 - \chi(p) \left(\frac{(-1)^{k-\frac{1}{2}} t}{p} \right) p^{k-\frac{3}{2}-s} \right] \cdot \left[1 - \omega_p p^{-s} + \chi(p)^2 p^{2k-2-2s} \right]^{-1}. \tag{21}$$

4.2 Shimura map and the Kohnen plus space

Shimura [Shi73] has shown given an Hecke eigenform f of $S(\Gamma_0(4N), k, j_{\Theta,\chi}^{2k})$ with $k \geq \frac{5}{2}$ one can use the Euler product (21) to construct a modular form $F(z)$ (22) in $S(\Gamma_0(M), 2k-1, j_{\chi^2}^{2k-1})$ for some M with the same eigenvalues. Here $j_{\chi^2}^{2k-1}$ denotes the automorphy factor $\chi(d)^2 j(\gamma, z)^{2k-1}$ on $\Gamma_0(M)$.

$$F(z) = \sum_{n=1}^{\infty} \hat{F}(n) e^{2\pi i n z}, \tag{22}$$

where

$$\sum_{n=1}^{\infty} \hat{F}(n) n^{-s} = \prod_p \left[1 - \omega_p p^{-s} + \chi(p)^2 p^{2k-2-2s}\right]^{-1}.$$

Niwa [Niw75] has shown, that M can be taken to be $2N$. By restriction to a certain subspace $S^+(\Gamma_0(4N), k, j_\Theta^{2k})$, the Kohnen plus space, Kohnen [Koh80], [Koh82] was able to reduce the level all the way down to N. The Kohnen plus space is given by:

$$S^+(\Gamma_0(4N), k, j_\Theta^{2k}) = \{f \in S(\Gamma_0(4N), k, j_\Theta^{2k}) | \hat{f}(n) = 0, \, \forall n : (-1)^{k-\frac{1}{2}} n \equiv 2, 3 \bmod (4)\}.$$

We now restrict ourselves to the simplest case $N = 1$, then the Shimura map restricts on the Kohnen plus space to an isomorphism of Hecke algebras, by replacing the Hecke operator $T(4)$ on $S^+(\Gamma_0(4), k, j_\Theta^{2k})$ by an other operator $T^+(4)$, which satisfies (20) also for $p = 2$ (i.e. without the second comment after the equation). As shown by Waldspurger and later made explicit by Kohnen and Zagier [KZ81] there is a correlation between values at the center of the critical strip of twists of the L-function associated to modular forms of the full group and the Fourier coefficients of their preimages under the Shimura map:

Theorem 4.2 (Kohnen-Zagier)**.** *Let $k \in \frac{1}{2} + \mathbb{Z}$, $F \in S(\mathrm{SL}_2(\mathbb{Z}), 2k-1, j^{2k-1})$ a normalized Hecke eigenform ($\hat{F}(1) = 1$), $f \in S^+(\Gamma_0(4), k, j_\Theta^{2k})$ a preimage of F under the Shimura map. Further let D be a fundamental discriminant with $(-1)^{k-\frac{1}{2}} D > 0$ and $L(F, \left(\frac{D}{\cdot}\right), s)$ the analytic continuation of the Dirichlet L-series $\sum_{n=1}^{\infty} \left(\frac{D}{n}\right) \frac{\hat{F}(n)}{n^{k-1}} n^{-s}$. Then*

$$\frac{|\hat{f}(|D|)|^2}{\langle f, f \rangle_{\Gamma_0(4)}} = \frac{\Gamma(k-\frac{1}{2})}{\pi^{k-\frac{1}{2}}} |D|^{k-1} \frac{L(F, \left(\frac{D}{\cdot}\right), \frac{1}{2})}{\langle F, F \rangle_{\mathrm{SL}_2(\mathbb{Z})}}.$$

Proof. We refer to [KZ81]. □

Remark 4.1. It is possible to normalize f such that all Fourier coefficients $\hat{f}(n)$ are real.

The other coefficients are related through (21) and (22):

$$\hat{f}(n^2|D|) = \hat{f}(|D|) \cdot \sum_{d|n} \mu(d) \left(\frac{D}{d}\right) d^{k-\frac{3}{2}} \hat{F}\left(\frac{n}{d}\right). \tag{23}$$

The Fourier coefficients at the other cusps $0, \frac{1}{2}$ are also related. For this we put:

$$W_4 = \left(\begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix}, (-2iz)^{\frac{1}{2}} \right), \quad V_4 = \left(\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, (-i(2z+1))^{\frac{1}{2}} \right),$$

And define a new operator U_4 given by

$$(f|U_4)(z) = \frac{1}{4} \sum_{i \bmod (4)} f\left(\frac{z+i}{4}\right) = \sum_{n\geq 1} \hat{f}(4n) e^{2\pi i n z}.$$

Lemma 4.3. *Let $k \in \frac{1}{2} + \mathbb{Z}, f \in S^+(\Gamma_0(4), k, j_\Theta^{2k})$. Then the Fourier coefficients of f at the cusps $0, \frac{1}{2}$ can be given in terms of the Fourier coefficients at ∞:*

$$\begin{aligned} (f|_k W_4)(z) &= \left(\frac{2}{2k}\right) 2^{\frac{1}{2}-k} \sum_{n\geq 1} \hat{f}(4n) e^{2\pi i n z}, \\ (f|_k V_4)(z) &= \left(\frac{2}{2k}\right) 2^{\frac{1}{2}-k} \sum_{\substack{n\geq 1, \\ (-1)^{k-\frac{1}{2}} n \equiv 1 \bmod (4)}} i^{\frac{n}{2}} \hat{f}(n) e^{2\pi i \frac{nz}{4}}. \end{aligned} \tag{24}$$

Note that $\left(\frac{2}{2k}\right)$ denotes the Jacobi symbol here.

Proof. In [Koh80] Prop. 2 Kohnen showed: $(f|U_4|_k W_4)(z) = \left(\frac{2}{2k}\right) 2^{k-\frac{1}{2}} f(z)$. Applying "$|_k W_4$" to both sides gives the desired result, by noting that "$|_k W_4^2$" is the identity map. The second identity follows from:

$$\begin{aligned} (f|_k V_4)(z) &= (-i(2z+1))^{-k} \sum_{\substack{n\geq 1, \\ (-1)^{k-\frac{1}{2}} n \equiv 0,1 \bmod (4)}} \hat{f}(n) e^{2\pi i n \left(\frac{1}{2} - \frac{1}{4z+2}\right)} \\ &= (-i(2z+1))^{-k} \left[2 \sum_{\substack{n\geq 1, \\ n\equiv 0 \bmod (4)}} - \sum_{\substack{n\geq 1, \\ (-1)^{k-\frac{1}{2}} n \equiv 0,1 \bmod (4)}} \right] \hat{f}(n) e^{2\pi i n \left(\frac{-1}{4z+2}\right)} \\ &= (-i(2z+1))^{-k} 2 (f|U_4) \left(\frac{-1}{z+\frac{1}{2}}\right) - (f|_k W_4)\left(z + \frac{1}{2}\right) \\ &= 4^{\frac{1}{2}-k} (f|U_4|_k W_4) \left(\frac{z+\frac{1}{2}}{4}\right) - (f|_k W_4)\left(z+\frac{1}{2}\right) \\ &= \left(\frac{2}{2k}\right) 2^{\frac{1}{2}-k} \left[f\left(\frac{z+\frac{1}{2}}{4}\right) - (f|U_4)\left(z+\frac{1}{2}\right)\right] \\ &= \left(\frac{2}{2k}\right) 2^{\frac{1}{2}-k} \sum_{\substack{n\geq 1, \\ (-1)^{k-\frac{1}{2}} n \equiv 1 \bmod (4)}} i^{n/2} \hat{f}(n) e^{2\pi i \frac{nz}{4}}. \end{aligned}$$

□

4.3 Sup-norm of modular forms in the Kohnen plus space

Assume $k \geq \frac{5}{2}$. Let $f \in S^+(\Gamma_0(4), k, j_\Theta^{2k})$ be a Hecke eigenform normalized with respect to the Petersson inner product. And denote by $F \in S(\Gamma, 2k-1, j^{2k-1})$ its image under the Shimura Map. Without loss of generality we assume, that F is a normalized Hecke eigenform ($\hat{F}(1) = 1$) and thus we can make use of Deligne's bound on the Fourier coefficients:

$$|\hat{F}(n)| \leq d(n)n^{k-1} \ll_\epsilon n^{k-1+\epsilon}.$$

As in the real weight case we want to bound $y^{\frac{k}{2}}|f(z)|, y^{\frac{k}{2}}|(f|_k W_4)(z)|$ and $y^{\frac{k}{2}}|(f|_k V_4)(z)|$. It is easily seen that for $y \geq \frac{\sqrt{3}}{8}$ they cover a fundamental domain of $\Gamma_0(4)$. Thus we may assume $y \geq \frac{\sqrt{3}}{8}$ in any further calculations. We will follow the method of Xia [Xia07]. For this we need bounds on the Fourier coefficients, which we will get through Theorem 4.2 and the following two propositions.

Proposition 4.4. *Let $F \in S(\Gamma, 2k-1, j^{2k-1})$ be a normalized Hecke eigenform, then we have:*

$$\langle F, F\rangle = \frac{\Gamma(2k-1)}{2^{4k-3}\pi^{2k}} L(\mathrm{sym}^2 F, 1),$$

where $L(\mathrm{sym}^2 F, s)$ is the analytic continuation of

$$\prod_p \left(1-\alpha_p^2 p^{2-2k-s}\right)^{-1}\left(1-\alpha_p\overline{\alpha_p}p^{2-2k-s}\right)^{-1}\left(1-\overline{\alpha_p}^2 p^{2-2k-s}\right)^{-1} = \frac{\zeta(2s)}{\zeta(s)}\sum_{n=1}^{\infty}\frac{\hat{F}(n)^2}{n^{s+2k-2}}$$

and $\alpha_p, \overline{\alpha_p}$ are the solutions to $\alpha_p + \overline{\alpha_p} = \hat{F}(p)$, $\alpha_p\overline{\alpha_p} = p^{2k-2}$.

Proof. See [Ran39]. □

Proposition 4.5. *Let $F \in S(\Gamma, 2k-1, j^{2k-1})$ be a normalized Hecke eigenform, then we have:*

$$k^{-\epsilon} \ll_\epsilon L(\mathrm{sym}^2 F, 1) \ll_\epsilon k^\epsilon.$$

Proof. See page 41 equation 2.16 of [Mic07]. □

Let us assume we have a uniform bound for $L(F, \chi, \frac{1}{2})$, where F a primitive holomorphic cusp form with respect to the automorphy factor j^{2k-1} on the full modular group $\mathrm{SL}_2(\mathbb{Z})$ and χ a primitive character modulo q. Which is of the type

$$\left|L\left(F, \chi, \frac{1}{2}\right)\right| \ll k^\alpha q^\beta \tag{25}$$

for some fixed constants α, β. Then we are able to estimate the Fourier coefficients in the following way. Using (23) with Deligne's bound one finds:

$$|\hat{f}(n^2|D|)| \ll_\epsilon |\hat{f}(|D|)| \sum_{d|n} d^{k-\frac{3}{2}}\left(\frac{n}{d}\right)^{k-1+\epsilon} \ll_\epsilon |\hat{f}(|D|)| \left(n^2\right)^{\frac{k-1}{2}+\epsilon}. \tag{26}$$

This and the theorems relating the Fourier coefficients at fundamental discriminants then gives for $f \in S^+(\Gamma_0(4), k, j_\Theta^{2k})$ with $k \geq \frac{5}{2}$:

$$|\hat{f}(n)| \ll_\epsilon \frac{(4\pi)^{\frac{k}{2}}}{\Gamma(k)^{\frac{1}{2}}} k^{\frac{\alpha}{2}+\epsilon} n^{\frac{k-1+\beta}{2}}. \tag{27}$$

As in the real case we use Lemma 3.18 to estimate sums that arise from the Fourier expansions (24) at $\infty, 0, \frac{1}{2}$.

$$\begin{aligned} y^{\frac{k}{2}}|f(z)| &\ll_\epsilon \frac{y^{\frac{k}{2}}(4\pi)^{\frac{k}{2}}k^{\frac{\alpha}{2}+\epsilon}}{\Gamma(k)^{\frac{1}{2}}} S\left(\frac{k-1+\beta}{2}, 2\pi y, 1\right), \\ y^{\frac{k}{2}}|(f|_k W_4)(z)| &\ll_\epsilon \frac{y^{\frac{k}{2}}(4\pi)^{\frac{k}{2}}k^{\frac{\alpha}{2}+\epsilon}}{\Gamma(k)^{\frac{1}{2}}} S\left(\frac{k-1+\beta}{2}, 2\pi y, 1\right), \\ y^{\frac{k}{2}}|(f|_k V_4)(z)| &\ll_\epsilon \frac{\left(\frac{y}{4}\right)^{\frac{k}{2}}(4\pi)^{\frac{k}{2}}k^{\frac{\alpha}{2}+\epsilon}}{\Gamma(k)^{\frac{1}{2}}} S\left(\frac{k-1+\beta}{2}, \frac{2\pi y}{4}, 1\right). \end{aligned} \tag{28}$$

We have:

$$S\left(\frac{k-1+\beta}{2}, 2\pi y, 1\right) \ll (4\pi)^{-\frac{k}{2}+\frac{1}{2}-\frac{\beta}{2}} y^{-\frac{k}{2}-\frac{1}{2}-\frac{\beta}{2}} k^{\frac{k}{2}+\frac{\beta}{2}} e^{-\frac{k}{2}} \left(1 + yk^{-\frac{1}{2}}\right).$$

We thus get the following proposition.

Proposition 4.6. *Let $k \in \frac{1}{2} + \mathbb{Z}$, $f \in S^+(\Gamma_0(4), k, j_\Theta^{2k})$ a Hecke eigenform normalised with respect to the Petersson inner product. Assuming* (25) *holds, then we have for $y \geq \frac{\sqrt{3}}{8}$:*

$$\begin{aligned} y^{\frac{k}{2}}|f(z)| &\ll_\epsilon \frac{k^{\frac{1}{4}+\frac{\alpha}{2}+\frac{\beta}{2}+\epsilon}}{y^{\frac{1}{2}+\frac{\beta}{2}}} \left(1 + yk^{-\frac{1}{2}}\right), \\ y^{\frac{k}{2}}|(f|_k W_4)(z)| &\ll_\epsilon \frac{k^{\frac{1}{4}+\frac{\alpha}{2}+\frac{\beta}{2}+\epsilon}}{y^{\frac{1}{2}+\frac{\beta}{2}}} \left(1 + yk^{-\frac{1}{2}}\right), \\ y^{\frac{k}{2}}|(f|_k V_4)(z)| &\ll_\epsilon \frac{k^{\frac{1}{4}+\frac{\alpha}{2}+\frac{\beta}{2}+\epsilon}}{y^{\frac{1}{2}+\frac{\beta}{2}}} \left(1 + yk^{-\frac{1}{2}}\right). \end{aligned}$$

For $y \geq \frac{12k}{\pi}$ we can again improve this using Lemmata 3.18, 3.20 and following to get:

$$S\left(\frac{k-1+\beta}{2}, 2\pi y, 1\right) \ll (4\pi y)^{-\frac{k}{2}-\frac{1}{2}-\frac{\beta}{2}} k^{\frac{k}{2}+\frac{\beta}{2}} e^{-\frac{k}{2}} \left(1 + k^{\frac{1}{2}} e^{-\pi y}\right).$$

Which gives the following proposition.

Proposition 4.7. *Let $k \in \frac{1}{2}+\mathbb{Z}$, $f \in S^{+}(\Gamma_0(4), k, j_{\Theta}^{2k})$ a Hecke eigenform normalised with respect to the Petersson inner product. Assuming* (25) *then we have for $y \geq \frac{12k}{\pi}$:*

$$y^{\frac{k}{2}}|f(z)| \ll_{\epsilon} \frac{k^{\frac{1}{4}+\frac{\alpha}{2}+\frac{\beta}{2}+\epsilon}}{y^{\frac{1}{2}+\frac{\beta}{2}}}\left(1+k^{\frac{1}{2}}e^{-\pi y}\right),$$

$$y^{\frac{k}{2}}|(f|_k W_4)(z)| \ll_{\epsilon} \frac{k^{\frac{1}{4}+\frac{\alpha}{2}+\frac{\beta}{2}+\epsilon}}{y^{\frac{1}{2}+\frac{\beta}{2}}}\left(1+k^{\frac{1}{2}}e^{-\pi y}\right),$$

$$y^{\frac{k}{2}}|(f|_k V_4)(z)| \ll_{\epsilon} \frac{k^{\frac{1}{4}+\frac{\alpha}{2}+\frac{\beta}{2}+\epsilon}}{y^{\frac{1}{2}+\frac{\beta}{2}}}\left(1+k^{\frac{1}{2}}e^{-\frac{\pi y}{4}}\right).$$

So if we assume the Lindelöf hypothesis ($\alpha=\beta=\epsilon$) we get the following theorem.

Theorem 4.8. *Let $k \in \frac{1}{2}+\mathbb{Z}$, $k \geq \frac{5}{2}$, $f \in S^{+}(\Gamma_0(4), k, j_{\Theta}^{2k})$ a Hecke eigenform normalised with respect to the Petersson inner product. Assuming the Lindelöf hypothesis for the quadratic twists of L-functions associated to Hecke eigenforms of weight $2k-1$ on the full modular group $\mathrm{SL}_2(\mathbb{Z})$ with respect to the automorphy factor j^{2k-1} we have:*

$$\sup_{z\in\mathbb{H}} y^{\frac{k}{2}}|f(z)| \ll_{\epsilon} k^{\frac{1}{4}+\epsilon}.$$

Proof. For $y \geq \frac{12k}{\pi}$ we use Proposition 4.7 and for $\frac{\sqrt{3}}{8} \leq y \leq \frac{12k}{\pi}$ we use Proposition 4.6. □

For an unconditional result we use a subconvexity result of Michel and Venkatesh [MV10], from which we deduce (25) with $\alpha=\beta=\frac{1}{2}-\delta$ for some $\delta>0$.

Theorem 4.9. *Let $k \in \frac{1}{2}+\mathbb{Z}$, $k \geq \frac{5}{2}$, $f \in S^{+}(\Gamma_0(4), k, j_{\Theta}^{2k})$ a Hecke eigenform normalised with respect to the Petersson inner product. Then we have that the sup-norm is bounded by:*

$$\sup_{z\in\mathbb{H}} y^{\frac{k}{2}}|f(z)| \ll k^{\frac{1}{2}}.$$

Proof. For $y \leq k^{\frac{1}{2}-\epsilon}$ we use Proposition 3.22, for $y \geq \frac{12k}{\pi}$ we use Proposition 4.7 and for y in between we use Proposition 4.6, where ϵ is chosen suitably small. □

Remark 4.2. The convexity bound itself gives that the sup-norm is bounded by $k^{\frac{1}{2}+\epsilon}$. To further improve this result with the given methods one needs $\alpha+\beta<\frac{1}{2}$, which is likely to be a hard problem.

We now come to lower bounds. Individually we can prove the following:

Proposition 4.10. *Let $k \in \frac{1}{2}+\mathbb{Z}$, $f \in S^{+}(\Gamma_0(4), k, j_{\Theta}^{2k})$ a Hecke eigenform normalised with respect to the Petersson inner product and F the corresponding normalized Hecke eigenform through the Shimura map. Then we have:*

$$\sup_{z\in\mathbb{H}} y^{\frac{k}{2}}|f(z)| \gg_{\epsilon} k^{\frac{1}{4}-\epsilon} \sup_{\substack{D \text{ fund. disc.},\\ (-1)^{k-\frac{1}{2}}D>0}} \sqrt{\frac{L(F,(\frac{D}{\cdot}),\frac{1}{2})}{|D|}}.$$

And we also have trivially

$$\sup_{z\in\mathbb{H}} y^{\frac{k}{2}}|f(z)| \gg 1.$$

Proof. We use (18) on the $|D|$-th Fourier coefficient together with Theorem 4.2 and Propositions 4.4, 4.5. The first part of the proposition then follows, by choosing $y = \frac{k}{4\pi|D|}$. On the other side we have

$$\sup_{z\in\mathbb{H}} y^k|f(z)|^2 \gg \langle f, f\rangle_{\Gamma_0(4)} = 1.$$

□

For the square average lower bound we need the Fourier coefficients of the projection of the Poincaré series of $S(\Gamma_0(4), k, j_\Theta^{2k})$ down to the Kohnen plus space. They have been computed by Kohnen [Koh85].

Proposition 4.11 (Kohnen). *Let $k \in \frac{1}{2} + \mathbb{Z}, k \geq \frac{5}{2}$ and $m \in \mathbb{N}, (-1)^{k-\frac{1}{2}}m \equiv 0, 1 \bmod(4)$. Then the Poincaré series $G_I^+(\Gamma_0(4), k, j_\Theta^{2k}, z, m)$ are given by the Fourier expansion:*

$$G_I^+(\Gamma_0(4), k, j_\Theta^{2k}, z, m) = \sum_{\substack{n\geq 1,\\ (-1)^{k-\frac{1}{2}}n\equiv 0,1 \bmod(4)}} g_{k,m}(n)e^{2\pi inz},$$

with

$$g_{k,m}(n) = \frac{2}{3}\left[\delta_{m,n} + (-1)^{\left\lfloor\frac{k+\frac{1}{2}}{2}\right\rfloor}\pi\sqrt{2}\left(\frac{n}{m}\right)^{\frac{k-1}{2}}\sum_{c\geq 1} H_c(n,m)J_{k-1}\left(\frac{\pi}{c}\sqrt{nm}\right)\right].$$

And $H_c(n,m)$ is given by:

$$H_c(n,m) = (1-(-1)^{k-\frac{1}{2}}i)\left(1+\left(\frac{4}{c}\right)\right)\frac{1}{4c}\sum_{\substack{\delta \bmod(4c),\\ (\delta,4c)=1}}\left(\frac{4c}{\delta}\right)\left(\frac{-4}{\delta}\right)^k \exp\left(2\pi i\frac{n\delta+m\delta^{-1}}{4c}\right).$$

The Poincaré series satisfy

$$\langle f, G_I^+(\Gamma_0(4), k, j_\Theta^{2k}, \cdot, m)\rangle = \frac{\Gamma(k-1)}{\mu(\Gamma_0(4))(4\pi m)^{k-1}}\hat{f}(m) \quad \forall f \in S^+(\Gamma_0(4), k, j_\Theta^{2k}).$$

Proof. See [Koh85] proposition 4. □

Corollary 4.12. *Let $k \in \frac{1}{2}+\mathbb{Z}, k \geq \frac{5}{2}$ and $\{f_j\}$ be an orthonormal basis of $S^+(\Gamma_0(4), k, j_\Theta^{2k})$, then we have*

$$\sum_j |\hat{f}_j(m)|^2 = \frac{\mu(\Gamma_0(4))(4\pi m)^{k-1}}{\Gamma(k-1)}\cdot\frac{2}{3}\left[1+(-1)^{\left\lfloor\frac{k+\frac{1}{2}}{2}\right\rfloor}\pi\sqrt{2}\sum_{c\geq 1}H_c(m,m)J_{k-1}\left(\frac{\pi m}{c}\right)\right].$$

Proof. The same proof as for Corollary 3.15 applies. □

We use the same inequalities (18), (19) as in the real weight case. For $k \geq 20m + 1$ we can use Proposition A.5 again to get:

$$\begin{aligned}\sup_{\operatorname{Im} z = y} \sum_j y^k |f_j(z)|^2 &\geq y^k e^{-4\pi m y} \cdot \sum_j \left| \hat{f}_j(m) \right|^2 \\ &\geq y^k e^{-4\pi m y} \cdot \frac{\mu(\Gamma_0(4))(4\pi m)^{k-1}}{\Gamma(k-1)} \cdot \frac{2}{3} \left[1 - 4\pi \left(\frac{\pi}{2} \right)^{k-1} \frac{\zeta(k-1)}{\Gamma(k)} \right].\end{aligned} \tag{29}$$

Chosing $y = \frac{4\pi m}{k}$ and $m = 1$ we get the following theorem.

Theorem 4.13. *Let $k \in \frac{1}{2} + \mathbb{Z}, k \geq 21$ and $\{f_j\}$ be an orthonormal basis of $S^+(\Gamma_0(4), k, j_\Theta^{2k})$, then we have*

$$\sum_j \sup_{z \in \mathbb{H}} y^k |f_j(z)|^2 \geq \sup_{z \in \mathbb{H}} \sum_j y^k |f_j(z)|^2 \gg \mu(\Gamma_0(4)) k^{\frac{3}{2}} \times \left(1 - O\left(\left(\frac{2\pi e}{k} \right)^{k-1} \right) \right).$$

Remark 4.3. This last theorem is easily generalized uniformly to $S^+(\Gamma_0(4N), k, j_\Theta^{2k})$.

Remark 4.4. By considering the dimension of the space $S^+(\Gamma_0(4), k, j_\Theta^{2k})$ this also shows that for Hecke eigenforms the best uniform upper bound one can hope for is $k^{\frac{1}{4}}$.

A Bessel functions

Recall that the (modified) Bessel functions are given by

$$
\begin{aligned}
J_\rho(x) &= \sum_{m=0} \frac{(-1)^m \left(\frac{x}{2}\right)^{2m+\rho}}{\Gamma(m+1)\Gamma(m+\rho+1)}, \\
Y_\rho(x) &= \sin(\rho\pi)^{-1} \left[J_\rho(x)\cos(\rho\pi) - J_{-\rho}(x)\right], \\
I_\rho(x) &= \sum_{m=0} \frac{\left(\frac{x}{2}\right)^{2m+\rho}}{\Gamma(m+1)\Gamma(m+\rho+1)}, \\
K_\rho(x) &= \frac{\pi}{2} \sin(\rho\pi)^{-1} \left[I_{-\rho}(x) - I_\rho(x)\right].
\end{aligned}
$$

Theorem A.1. *On has for $x \geq C > 0$:*

$$
\begin{aligned}
|J_\rho(x)| &\ll_{C,\rho} x^{-\frac{1}{2}}, \\
|Y_\rho(x)| &\ll_{C,\rho} x^{-\frac{1}{2}}, \\
|K_\rho(x)| &\ll_{C,\rho} x^{-\frac{1}{2}} e^{-x}.
\end{aligned}
$$

Proof. See [Wat44] page 199, 202. □

Theorem A.2 (Langer's formulas)**.** *The Bessel function admits the following uniform formula for $x > \rho$:*

$$
J_\rho(x) = w^{-\frac{1}{2}}(w - \arctan w)^{\frac{1}{2}} \left[\frac{\sqrt{3}}{2} J_{\frac{1}{3}}(z) - \frac{1}{2} Y_{\frac{1}{3}}(z)\right] + O(\rho^{-\frac{4}{3}}),
$$

where

$$
w = \sqrt{\frac{x^2}{\rho^2} - 1} \text{ and } z = \rho(w - \arctan(w)).
$$

For $x < \rho$ one has the formula

$$
J_\rho(x) = \pi^{-1} w^{-\frac{1}{2}} (\operatorname{artanh}(w) - w)^{\frac{1}{2}} K_{\frac{1}{3}}(z) + O(\rho^{-\frac{4}{3}}),
$$

where

$$
w = \sqrt{1 - \frac{x^2}{\rho^2}} \text{ and } z = \rho(\operatorname{artanh}(w) - w).
$$

Proof. See [EMOT81] page 30, 89. □

Theorem A.3. *For the intermediate range $|x - \rho| = o(\rho^{\frac{1}{3}})$ we have the following asymptotic for every M:*

$$
J_\rho(x) = \frac{1}{3\pi} \sum_{m=0}^{M-1} B_m(x-\rho) \sin\left(\frac{\pi}{3}(m+1)\right) \frac{\Gamma\left(\frac{1}{3}(m+1)\right)}{\left(\frac{x}{6}\right)^{\frac{1}{3}(m+1)}} + O\left(x^{-\frac{M+1}{3}}\right).
$$

Proof. See [Wat44] page 245-247. □

The proof given there is also enough to show the following proposition.

Proposition A.4. *For $|x-\rho| \leq C\rho^{\frac{1}{3}}, \rho \gg_C 1$ we have the following:*

$$|J_\rho(x)| \ll_C \rho^{-\frac{1}{3}}.$$

Proposition A.5. *One has for $\rho \geq 2x^2$:*

$$|J_\rho(x)| \ll \frac{\left(\frac{x}{2}\right)^\rho}{\Gamma(\rho+1)}.$$

Proof. Using Stirling approximation for the Γ-function one checks that:

$$\begin{aligned}
|J_\rho(x)| &= \left|\sum_{m=0} \frac{(-1)^m \left(\frac{x}{2}\right)^{2m+\rho}}{\Gamma(m+1)\Gamma(m+\rho+1)}\right| \\
&\ll \frac{\left(\frac{x}{2}\right)^\rho}{\Gamma(\rho+1)} \sum_{m=0}^{\infty} \frac{(\rho+1)^{\rho+\frac{1}{2}}}{(m+1)^{m+\frac{1}{2}}(\rho+m+1)^{\rho+m+\frac{1}{2}}e^{-2m}} \left(\frac{x}{2}\right)^{2m} \\
&\ll \frac{\left(\frac{x}{2}\right)^\rho}{\Gamma(\rho+1)} \sum_{m=0}^{\infty} \left(\frac{\rho+1}{\rho+m+1}\right)^\rho \left(\frac{x^2e^2}{4(m+1)(\rho+m+1)}\right)^m \\
&\ll \frac{\left(\frac{x}{2}\right)^\rho}{\Gamma(\rho+1)} \sum_{m=0}^{\infty} \left(\frac{x^2e^2}{4(\rho+1)}\right)^m \\
&\ll \frac{\left(\frac{x}{2}\right)^\rho}{\Gamma(\rho+1)}.
\end{aligned}$$

□

Proposition A.6. *There exists $C' > 0$ for which we have in the range $x \leq \rho - C'\rho^{\frac{1}{3}}(\log\rho)^{\frac{1}{3}}, \rho \gg_{C'} 1$ the following estimation:*

$$|J_\rho(x)| \ll_{C'} \rho^{-\frac{4}{3}}.$$

Proof. We are going to use Langer's formula (see Theorem A.2) for $x < \rho$. There $z = \rho\sum_{n=1}^{\infty}\frac{w^{2n+1}}{2n+1} \geq \log\rho$ for a particular choice of C' and we estimate by using Theorem A.1:

$$\begin{aligned}
|J_\rho(x)| &= |\pi^{-1}w^{-\frac{1}{2}}(\operatorname{artanh}(w)-w)^{\frac{1}{2}}K_{\frac{1}{3}}(z)| + O(\rho^{-\frac{4}{3}}) \\
&\ll_{C'} (\rho w)^{-\frac{1}{2}}e^{-z} + O(\rho^{-\frac{4}{3}}) \\
&\ll_{C'} \left(2C'\rho^{\frac{4}{3}}(\log\rho)^{\frac{1}{3}} - C'^2\rho^{\frac{2}{3}}(\log\rho)^{\frac{2}{3}}\right)^{-\frac{1}{4}} \rho^{-1} + O(\rho^{-\frac{4}{3}}) \\
&\ll_{C'} \rho^{-\frac{4}{3}}.
\end{aligned}$$

□

A similar argument can be applied to get the next proposition.

Proposition A.7. *We have for the range* $\rho - C\rho^{\frac{1}{3}} \geq x \geq \rho - C\rho^{\frac{1}{3}}(\log\rho)^{\frac{1}{3}}, \rho \gg_C 1$ *the following estimation:*

$$|J_\rho(x)| \ll_C \rho^{-\frac{1}{3}}.$$

Proposition A.8. *For* $x \geq \rho + C\rho^\alpha, \rho \gg_{C,\alpha} 1$ *we have:*

$$|J_\rho(x)| \ll_C \begin{cases} \rho^{-\frac{\alpha+1}{4}}, & \textit{for } \frac{1}{3} \leq \alpha \leq 1, \\ x^{-\frac{1}{2}} \ll_C \rho^{-\frac{\alpha}{2}}, & \textit{for } 1 \leq \alpha \leq \frac{8}{3}, \\ \rho^{-\frac{4}{3}}, & \textit{for } \frac{8}{3} \leq \alpha. \end{cases}$$

Proof. Use Langer's formula (see Theorem A.2) for $x > \rho$. And note that for $\alpha \geq \frac{1}{3}$ and $\rho \gg_{C,\alpha} 1$ we have $z \gg 1$. We can thus use Theorem A.1 to deduce:

$$\begin{aligned} |J_\rho(x)| &= \left| w^{-\frac{1}{2}}(w - \arctan w)^{\frac{1}{2}} \left[\frac{\sqrt{3}}{2} J_{\frac{1}{3}}(z) - \frac{1}{2} Y_{\frac{1}{3}}(z) \right] \right| + O(\rho^{-\frac{4}{3}}) \\ &\ll_C (\rho w)^{-\frac{1}{2}} + O(\rho^{-\frac{4}{3}}) \\ &\ll_C (x^2 - \rho^2)^{-\frac{1}{4}} + O(\rho^{-\frac{4}{3}}). \end{aligned}$$

From which the proposition follows. □

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