THE HARMONIC CONJUNCTION OF AUTOMORPHIC FORMS AND THE HARDY–LITTLEWOOD CIRCLE METHOD

RAPHAEL S. STEINER

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Raphael S. Steiner:

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ABSTRACT

The delta symbol circle method is applied to the problem of equidistribution of rational points on shrinking sets of the 3-sphere. This leads to a correlation sum of Kloosterman sums and an exponential function also known as the twisted Linnik–Selberg conjecture, which is further analysed by means of the Kuznetsov trace formula. Furthermore, the circle method in the guise of Wooley's efficient congruencing machinery is employed in an effective manner to obtain effective bounds on Vinogradov's mean value theorem.

To all those who have inspired me.

If you want to build a ship,
don't drum up people to collect wood and don't assign them tasks and work,
but rather teach them to long for the endless immensity of the sea.

— Antoine de Saint-Exupéry

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DECLARATION

I declare that the work in this dissertation was carried out in accordance with the requirements of the University's 'Regulations and Code of Practice for Research Degree Programmes' and that it has not been submitted for any other academic award. Except where indicated by specific reference in the text, the work is my own work. Work done in collaboration with, or with the assistance of, others is indicated as such. Any views expressed in this dissertation are those of the author.

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1

INTRODUCTION

In this thesis, the interplay between automorphic forms and the Hardy–Littlewood circle method is explored. Whilst these two subjects now appear to have little in common, this was not always so. Just over a century ago, the circle method was born and first applied to an automorphic form, namely, the Dedekind eta function, in the works of Hardy and Ramanujan [HR18], who were interested in an asymptotic for the partition numbers. They found that its generating series, which is related to the Dedekind eta function, becomes large as the argument approached the unit circle at an angle $e^{2\pi i r}$ for rational r with small denominator and is small otherwise. Thus, they concluded that the growth of the partition numbers is dominated by the behaviour of the Dedekind eta function at those rational numbers r. It proved useful that the Dedekind eta function satisfies modular relations, being an automorphic form, which facilitated a simple description of its behaviour at those rationals. In the 1920's, Hardy and Littlewood started to apply the same philosophy to other problems, such as Waring's problem [HL20], with great success. Nowadays, the circle method takes on many forms, with contributions having been made by various authors; notably Vinogradov, who introduced his eponymous mean value theorem [Vin35b, Vin35a], with far-reaching application to exponential sums. This is further discussed in Chapter 5; in particular, in Section 5.3, where we shall prove an effective Vinogradov mean value theorem based on new developments by Wooley [Woo12].

The two subjects overlap further in the theory of quadratic forms. The connection here is that the generating function is given by a theta function, which is in turn an automorphic form. Furthermore, the exponential sums that crop up in connection with quadratic forms, Kloosterman and Salié sums, are also of significance to both subjects. Originally, they were discovered by Poincaré as part of the Fourier coefficients of the Poincaré series, yet another set of important automorphic forms, but only gained (at)traction after they appeared in Kloosterman's refinement of the circle method. Shortly thereafter, they were also used by Kloosterman [Klo27] to give bounds on the size of the Fourier coefficients of holomorphic forms. Today, optimal bounds on the size of the Fourier coefficients of holomorphic forms (of integral weight), as well as the size of Kloosterman sums, are known from arithmetic geometry. Nonetheless, there remain many open questions concerning Kloosterman sums. One of these questions is the Linnik–Selberg Conjecture on sums of Kloosterman sums. The only non-trivial progress towards this conjecture stems from Kuznetsov's trace formula [Kuz80], which we shall prove in great generality in Section 3.10. In Chapter 4, we shall apply the Kuznetsov trace formula to a twisted version of the Linnik–Selberg Conjecture on sums of Kloosterman sums, which resurfaces in Chapter 6. The final part of this thesis concerns a problem about quadratic forms. Concretely, we shall be looking at the solutions to the equation $x_1^2 + x_2^2 + x_3^2 + x_4^2 = N$ and how fast their projections onto S^3 equidistribute with respect to the Lebesgue measure as N runs over the odd integers. This has far-reaching consequences to efficient quantum computing on 1-qubits. We shall employ, compare, and contrast an automorphic and a circle-method approach to this problem in Chapter 6.

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NOTATION

The set of natural numbers is denoted by $\mathbb{N}$, which start at 1. If 0 is to be included in this set it is denoted by $\mathbb{N}_0$. The set of integers is denoted by $\mathbb{Z}$. The symbols $\mathbb{Q}, \mathbb{R}$ denote the set of rational numbers, respectively the set of real numbers. The subset of (strictly) positive rational, respectively real, numbers is denoted by $\mathbb{Q}^+$, respectively $\mathbb{R}^+$. If 0 is to be included in these sets, then they are denote by $\mathbb{Q}_0^+$, respectively $\mathbb{R}_0^+$, which now consist of all non-negative rational, respectively real, numbers. Closed intervals are denoted by $[a, b]$, open ones by $]a, b[$, and half-open ones by $[a, b[$, respectively $]a, b]$. The characteristic function of an interval $\mathcal{I}$ is denoted by $\mathbb{1}_{\mathcal{I}}$. The set of complex numbers is denoted by $\mathbb{C}$. The letter s is usually used to denote a complex number. The real part and the imaginary part of a complex number s are denoted by $\mathrm{Re}(s)$ and $\mathrm{Im}(s)$, respectively. The complex conjugate of s is denoted by $\overline{s}$, the norm of s by $|s| = \sqrt{s\overline{s}}$, and the argument of $s \neq 0$ by $\arg(s)$. The argument shall always denote its principal value, i.e. $-\pi < \arg(s) \leq \pi$. The subset of the complex numbers with positive imaginary part is denoted by $\mathbb{H}$ and is referred to as the upper half-plane. A complex number in the upper half-plane is often denoted by $z = x + iy$, where x denotes the real part of z and y its imaginary part. Vectors are emphasised using bold font, for example $\boldsymbol{v}, \boldsymbol{w}$, and inner products are either denoted by $\langle \boldsymbol{v}, \boldsymbol{w} \rangle$ or by $\boldsymbol{v} \cdot \boldsymbol{w}$.

The Möbius function is denoted by μ. The divisor function is denoted by τ. The number of representations of n as a sum of m squares of integers is denoted by $r_m(n)$. The Jacobi symbol is denoted by $(\frac{m}{n})$. The prime in $\sum'_{a \bmod (c)}$ indicates that the sum is restricted to those $a \bmod (c)$ with $(a, c) = 1$. The function $\exp(2\pi i z)$ is usually abbreviated to $e(z)$ and sometimes $e_q(z)$ is used to denote $e(\frac{z}{q})$. For a non-zero complex number $s \in \mathbb{C}^\times$, the logarithm $\log(s)$ denotes its principal value, i.e. $\log(s) = \log(|s|) + i\arg(s)$, and exponentiating by another complex number w is defined as $s^w = \exp(w \cdot \log(s))$. The meromorphic extension of the Gamma function is denoted by $\Gamma(s)$. L_p-norms are denoted by $\|\cdot\|_p$ and $\|\cdot\|_{M,p}$ denotes the Sobolev L_p-norm of order M. The integral $\int_{-\infty}^{(0^+)}$ denotes a lock-hole-shaped contour integral also known as a Hankel contour, which starts at $-\infty$, loops around 0 once in positive direction (anti-clockwise), and goes back to $-\infty$. It will often be the case for this type of integral that the logarithm takes its principal value, except for the non-positive real axis, where it is two-valued in a continuous manner.

We further adopt Landau's big O notation and Vinogradov's notations $\ll$, $\asymp$. The latter is used out of convenience of the notation, where $f \ll g$ has the same meaning as $f = O(g)$ and $f \asymp g$ has the same meaning as $f = \Theta(g)$, i.e. $f \ll g \ll f$. Subscripts, such as $O_{A,B}$ or $\ll_{A,B}$, mean that the implied constant may depend on A and B. A subscript of ϵ, for example $f \ll_\epsilon x^\epsilon$, however shall mean for every sufficiently small $\epsilon > 0$ there is a constant C, which may depend on ϵ, such that we have $|f| \leq Cx^\epsilon$. Occasionally, ϵ is used otherwise in which case we shall abuse some notation and write $\ll x^{o(1)}$, which shall mean $\ll_\epsilon x^\epsilon + x^{-\epsilon}$ (in the above mentioned sense).

3

AUTOMORPHIC FORMS

3.1 INTRODUCTION

The theory of automorphic forms finds its origins in elliptic functions and was generalised by F. Klein and others to a more abstract setting. Let X be a Riemannian manifold and Γ a group acting properly discontinuously on X. Then, a function that is invariant under the action of Γ and also an eigenfunction of all invariant differential operators on X is called an automorphic function of $\Gamma \setminus X$. More generally, one can allow for certain multipliers, describing the way in which Γ acts on $C^\infty(X, \mathbb{R})$. Here, we are concerned with the special case $X = \mathrm{SL}_2(\mathbb{R}) \,/\, \mathrm{SO}_2(\mathbb{R})$. Maass [Maa49] discovered a connection between the automorphic forms for this particular example and degree two L-functions and henceforth they carry his name. Much of the theory we know today was worked out by Maass [Maa52] and Selberg [Sel89]. The theory, which we summarise here, is mostly based on Roelcke [Roe66], Iwaniec [Iwa97, Iwa02], and Rankin [Ran77], and citations are given whenever possible. However, in this thesis, we consider the subject in slightly more generality and a reference is not always readily available. Due to some subtleties, e.g. branches of logarithms, proofs are carried out in those cases, despite being of similar nature to proofs in a more classical setting.

3.2 FUCHSIAN GROUPS OF THE FIRST KIND

The upper half-plane $\mathbb{H}$ may be identified as the quotient $\mathrm{SL}_2(\mathbb{R}) \,/\, \mathrm{SO}_2(\mathbb{R})$ via the map $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto (ai+b)(ci+d)^{-1}$ with inverse given by $x + iy \mapsto \frac{1}{\sqrt{y}} \left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right)$. This gives rise to a natural action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$, which is given by

$$\gamma \cdot z = \gamma z = \frac{az+b}{cz+d}, \text{ where } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R}).$$

This map is also known as a Möbius transformation. This action can be further extended to the real projective line $\mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\} =: \partial\mathbb{H}$. The cocycle $j : \mathrm{SL}_2(\mathbb{R}) \times \mathbb{H} \to \mathbb{C}^\times$ given by

$$j(\gamma, z) = cz + d, \text{ where } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R}), z \in \mathbb{H},$$

is related to the Möbius transformation $z \mapsto \gamma z$, say its derivative is $j(\gamma, z)^{-2}$ and $\mathrm{Im}(\gamma z) = \mathrm{Im}(z)/|j(\gamma, z)|^2$, and will appear rather frequently.

Definition 3.2.1. A subgroup $\hat{\Gamma} \subseteq \mathrm{PSL}_2(\mathbb{R})$ is called a *Fuchsian subgroup of the first kind* if $\hat{\Gamma}$ acts discontinuously on $\mathbb{H}$ and every point in $\partial\mathbb{H}$ is a limit point of $\hat{\Gamma}z$ for some $z \in \mathbb{H}$.

For a Fuchsian group $\hat{\Gamma}$ of the first kind, we let Γ be the pre-image of $\hat{\Gamma}$ under the projection $\mathrm{SL}_2(\mathbb{R}) \to \mathrm{PSL}_2(\mathbb{R})$. Vice versa, for $\Gamma \subseteq \mathrm{SL}_2(\mathbb{R})$ with $-I \in \Gamma$ we let $\hat{\Gamma}$ be the image of Γ. Moreover, we say that Γ is a Fuchsian group of the first kind if $\hat{\Gamma}$ is.

Example 3.2.1. Typical examples include $\mathrm{PSL}_2(\mathbb{Z})$, congruence subgroups $\hat{\Gamma}_0(N)$, $\hat{\Gamma}_1(N)$, $\hat{\Gamma}(N)$, and the theta subgroup $\hat{\Gamma}_\theta$. Here,

$$\begin{aligned}
\hat{\Gamma}_0(N) &= \left\{\gamma \in \mathrm{PSL}_2(\mathbb{Z}) | \gamma \equiv \left(\begin{smallmatrix} \star & \star \\ 0 & \star \end{smallmatrix}\right) \bmod(N)\right\}, \\
\hat{\Gamma}_1(N) &= \left\{\gamma \in \mathrm{PSL}_2(\mathbb{Z}) | \gamma \equiv \pm \left(\begin{smallmatrix} 1 & \star \\ 0 & 1 \end{smallmatrix}\right) \bmod(N)\right\}, \\
\hat{\Gamma}(N) &= \left\{\gamma \in \mathrm{PSL}_2(\mathbb{Z}) | \gamma \equiv \pm I \bmod(N)\right\}, \\
\hat{\Gamma}_\theta &= \left\{\gamma \in \mathrm{PSL}_2(\mathbb{Z}) | \gamma \equiv I \text{ or } \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \bmod(2)\right\}.
\end{aligned}$$

Definition 3.2.2. Let $\hat{\Gamma}$ be a Fuchsian group of the first kind. A domain $F \subseteq \mathbb{H}$ is called a *fundamental domain* for $\hat{\Gamma}$ if

1. $\forall z, w \in F : \hat{\Gamma}z \cap \hat{\Gamma}w \neq \emptyset \Leftrightarrow z = w$,
2. $\forall z \in \mathbb{H} : \hat{\Gamma}z \cap \overline{F} \neq \emptyset$.

We shall often denote such a set F by $\mathcal{F}_\Gamma$.

A fundamental domain of a Fuchsian group of the first kind need not be compact. It may contain segments which diverge to points in $\partial\mathbb{H}$. We call such points cusps. Formally, we shall define a cusp as follows.

Definition 3.2.3. Let $\hat{\Gamma}$ be a Fuchsian group of the first kind. A point $\mathfrak{a} \in \partial\mathbb{H}$ is called a *cusp* of $\hat{\Gamma}$ if it is fixed by a parabolic element of $\hat{\Gamma}$. We say two cusps $\mathfrak{a}, \mathfrak{b}$ are *equivalent* if $\hat{\Gamma}\mathfrak{a} = \hat{\Gamma}\mathfrak{b}$ and the set of equivalence classes of cusps we refer to as the cusps of $\hat{\Gamma}$.

From here on onwards, we shall fix one cusp for each equivalence class of cusps. This will simplify matters as not everything we shall define will be independent of the choice of representative of an equivalence class.

The stabiliser group $\hat{\Gamma}_{\mathfrak{a}}$ of a cusp $\mathfrak{a}$ is cyclic. This follows since $\hat{\Gamma}_{\mathfrak{a}}$ is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})_{\mathfrak{a}}$, which is a one-parameter subgroup. To this end let $\hat{\gamma}_{\mathfrak{a}}$ be a generator. We may wish to translate the cusp $\mathfrak{a}$ to ∞, which we achieve through a scaling matrix.

Definition 3.2.4. Let $\mathfrak{a}$ be a cusp of $\hat{\Gamma}$. A matrix $\sigma_{\mathfrak{a}} \in \mathrm{SL}_2(\mathbb{R})$ is called *scaling matrix* for $\mathfrak{a}$ if it satisfies the following properties:

1. $\sigma_{\mathfrak{a}}\infty = \mathfrak{a}$,
2. $\hat{\sigma}_{\mathfrak{a}}^{-1}\hat{\gamma}_{\mathfrak{a}}\hat{\sigma}_{\mathfrak{a}} = \pm\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$,
3. $\sigma_{\mathfrak{a}} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with either $c \neq 0$ or $c = 0$ and $d > 0$.

Remark 3.2.2. The last condition is not necessary and is usually omitted. However, we include it as it implies $\sigma_\kappa(\sigma_{\mathfrak{a}}, \sigma_{\mathfrak{a}}^{-1}) = \sigma_\kappa(\sigma_{\mathfrak{a}}^{-1}, \sigma_{\mathfrak{a}}) = 1$ (see (3.1) for the definition of σ_κ).

Proposition 3.2.3. *Every Fuchsian group of the first kind has a fundamental domain of finite volume and the volume depends only on the group itself.*

Proof. See [Sie43]. □

Next, we shall define what constitutes an automorphy factor. Although, there is a notion of automorphy factor of complex weight $\kappa \in \mathbb{C}$, we shall restrict ourselves to real weight $\kappa \in \mathbb{R}$.

Definition 3.2.5. An *automorphy factor* ν of weight κ with respect to Γ is a function $\nu : \Gamma \times \mathbb{H} \to \mathbb{C}$ that satisfies the properties

1. $\forall \gamma, \tau \in \Gamma, \forall z \in \mathbb{H} : \nu(\gamma\tau, z) = \nu(\gamma, \tau z)\nu(\tau, z)$,
2. $\forall \gamma \in \Gamma, \forall z \in \mathbb{H} : |\nu(\gamma, z)| = |j(\gamma, z)|^\kappa$,
3. $\forall z \in \mathbb{H} : \nu(-I, z) = 1$.

Given an automorphy factor ν of weight κ we can define an associated multiplier system $\upsilon : \Gamma \to S^1$ by $\nu(\gamma, z) = \upsilon(\gamma)j(\gamma, z)^\kappa$ for $\gamma \in \Gamma$ and $z \in \mathbb{H}$. Note that υ is indeed independent of z by the maximum modulus principle. In order to quantify the relations a multiplier system must satisfy to be associated to an automorphy factor, we need to introduce a correction factor, which is given by

$$\sigma_\kappa(\gamma,\tau)=\frac{j(\gamma,\tau z)^\kappa j(\tau,z)^\kappa}{j(\gamma\tau,z)^\kappa},\quad \forall\gamma,\tau\in\mathrm{SL}_2(\mathbb{R}),\forall z\in\mathbb{H}. \tag{3.1}$$

This is again independent of z by the maximum modulus principle. We are now able to give the defining properties of a multiplier system.

Definition 3.2.6. A *multiplier system* υ of weight κ with respect to Γ is a function $\upsilon:\Gamma\to S^1$ that satisfies the properties

1. $\forall\gamma,\tau\in\Gamma:\upsilon(\gamma\tau)=\upsilon(\gamma)\upsilon(\tau)\sigma_\kappa(\gamma,\tau)$,
2. $\upsilon(-I)=e\left(-\frac{\kappa}{2}\right)$.

Note that, if υ is a multiplier system of weight κ for Γ, then it is also one for every weight in $\kappa+2\mathbb{Z}$. Moreover, $\overline{\upsilon}$ is a multiplier system of weight $-\kappa$ for Γ. The behaviour of a multiplier system at a cusp is going to be of significance.

Definition 3.2.7. Let υ be a multiplier system for Γ and $\mathfrak{a}$ be a cusp of Γ. Then, the *cusp parameter* $\eta_\mathfrak{a}^\upsilon=\eta_\mathfrak{a}$ is defined by $\upsilon(\sigma_\mathfrak{a}\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\sigma_\mathfrak{a}^{-1})=e(\eta_\mathfrak{a})$ and $\eta_\mathfrak{a}\in[0,1[$. A cusp $\mathfrak{a}$ is said to be *singular* with respect to υ if $\eta_\mathfrak{a}=0$.

For technical reason we let $\delta_\mathfrak{a}^{ns}$ denote the indicator function for non-singular cusps. Note that we have $\delta_\mathfrak{a}^{ns}=\eta_\mathfrak{a}^\upsilon+\eta_\mathfrak{a}^{\overline{\upsilon}}$.

We shall further require some transformation laws of the correction factor σ_κ.

Lemma 3.2.4. *The following relations are valid for* $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)=\gamma_1,\gamma_2,\gamma_3\in\mathrm{SL}_2(\mathbb{R})$ *and* $\forall\rho\in\left(\begin{smallmatrix}\mathbb{R}^+&\mathbb{R}\\\{0\}&\mathbb{R}^+\end{smallmatrix}\right)\cap\mathrm{SL}_2(\mathbb{R})$*:*

$$\sigma_\kappa(\gamma_1,\gamma_2\gamma_3)\sigma_\kappa(\gamma_2,\gamma_3)=\sigma_\kappa(\gamma_1,\gamma_2)\sigma_\kappa(\gamma_1\gamma_2,\gamma_3), \tag{3.2}$$

$$\sigma_\kappa(\gamma_1,\rho)=\sigma_\kappa(\rho,\gamma_1)=1, \tag{3.3}$$

$$\sigma_\kappa(\gamma_1,\gamma_2)=\sigma_\kappa(\rho\gamma_1,\gamma_2)=\sigma_\kappa(\gamma_1,\gamma_2\rho), \tag{3.4}$$

$$\sigma_\kappa(\gamma_1\rho,\gamma_2)=\sigma_\kappa(\gamma_1,\rho\gamma_2), \tag{3.5}$$

$$\sigma_\kappa(\gamma_1\rho\gamma_1^{-1},\gamma_1)=\sigma_\kappa(\gamma_1,\gamma_1^{-1}\rho\gamma_1)=1, \tag{3.6}$$

$$\sigma_\kappa(\gamma_1\rho\gamma_1^{-1},\gamma_1\gamma_2^{-1})=\sigma_\kappa(\gamma_1\gamma_2^{-1},\gamma_2\rho\gamma_2^{-1}), \tag{3.7}$$

(3.8)

$$\sigma_\kappa(\gamma_1, \gamma_1^{-1}) = \sigma_\kappa(\gamma_1^{-1}, \gamma_1) = \begin{cases} 1, & \text{if } c \neq 0 \text{ or } c = 0 \text{ and } d > 0, \\ e(\kappa), & \text{if } c = 0 \text{ and } d < 0. \end{cases} \tag{3.9}$$

Proof. See [Ran77, Chapter 3]. □

Proposition 3.2.5. *Given a matrix $\tau \in \mathrm{SL}_2(\mathbb{R})$ and a multiplier system υ of weight κ on Γ we can define a conjugate multiplier system υ^τ on $\tau^{-1}\Gamma\tau$ by*

$$\upsilon^\tau(\gamma) = \upsilon(\tau\gamma\tau^{-1})\frac{\sigma_\kappa(\tau\gamma\tau^{-1}, \tau)}{\sigma_\kappa(\tau, \gamma)} \quad \forall \gamma \in \tau^{-1}\Gamma\tau.$$

Proof. This is easily verified using Lemma 3.2.4 or see [Ran77, pp. 72-73]. □

Proposition 3.2.6. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two cusp for Γ. Then, we have a decomposition*

$$\sigma_{\mathfrak{a}}^{-1}\hat{\Gamma}\sigma_{\mathfrak{b}} = \delta_{\mathfrak{a},\mathfrak{b}} B \sqcup \bigsqcup_{c>0} \bigsqcup_{d \bmod c\mathbb{Z}} B\omega_{c,d}B,$$

where $B = \{\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) | b \in \mathbb{Z}\}$ and the union is only taken over those pairs (c, d) for which there exist a matrix $\left(\begin{smallmatrix} \star & \star \\ c & d \end{smallmatrix}\right) \in \sigma_{\mathfrak{a}}^{-1}\hat{\Gamma}\sigma_{\mathfrak{b}}$, and $\omega_{c,d}$ denotes an arbitrarily chosen one thereof. The set of these c's will be of importance and we shall denote it by $\mathcal{C}_{\mathfrak{a},\mathfrak{b}}$, i.e.

$$\mathcal{C}_{\mathfrak{a},\mathfrak{b}} = \left\{ c \in \mathbb{R}^+ \middle| \exists \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}} \right\}.$$

Proof. This is [Iwa02, Theorem 2.7]. □

Lemma 3.2.7. *We have $\mathcal{C}_{\mathfrak{a},\mathfrak{b}} = \mathcal{C}_{\mathfrak{b},\mathfrak{a}}$.*

Proof. The map $\gamma \mapsto -\gamma^{-1}$ is clearly an involution from $\sigma_{\mathfrak{a}}^{-1}\hat{\Gamma}\sigma_{\mathfrak{b}} \to \sigma_{\mathfrak{b}}^{-1}\hat{\Gamma}\sigma_{\mathfrak{a}}$, which induces an involution $\mathcal{C}_{\mathfrak{a},\mathfrak{b}} \to \mathcal{C}_{\mathfrak{b},\mathfrak{a}}$ given by $c \mapsto c$. □

Let us denote by $c_{\mathfrak{a},\mathfrak{b}}$ the smallest element of $\mathcal{C}_{\mathfrak{a},\mathfrak{b}}$ and simply $c_{\mathfrak{a}}$ for $c_{\mathfrak{a},\mathfrak{a}}$. The existence of such a minimal element is proven [Iwa02][Section 2.6], however the proof given there requires some knowledge on the geometry of Fuchsian groups. We give a simpler proof here. Let us consider the special case $\mathfrak{a} = \mathfrak{b}$ first. Suppose $\mathcal{C}_{\mathfrak{a},\mathfrak{a}}$ is the empty set. Then, by using Proposition 3.2.6 we have $\sigma_{\mathfrak{a}}^{-1}\hat{\Gamma}\sigma_{\mathfrak{a}} = B$ which implies that a fundamental domain of Γ has infinite volume, a contradiction. Thus, we may define $c_{\mathfrak{a}}$ as the infimum.

We wish to show $c_{\mathfrak{a}} > 0$. By invoking [Iwa02][Prop. 2.1] we know that Fuchsian subgroups are discrete subgroups of $\mathrm{SL}_2(\mathbb{R})$. Let $\gamma = \left(\begin{smallmatrix} \star & \star \\ c & \star \end{smallmatrix}\right) \in \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{a}}$. Then, we may find $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \pm B\gamma B \subseteq \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{a}}$ with $a = 1 + O(c)$, $b = O(1)$, and $d = 1 + O(c)$. If there were arbitrarily small such c's we would reach a contradiction with the discreteness of Γ. Now, Proposition 3.3.3 shows that the set $\mathcal{C}_{\mathfrak{a},\mathfrak{a}}$ must be discrete. Hence $c_{\mathfrak{a}}$ is indeed a minimum. The general case follows from Propositions 3.2.6, which shows that $\mathcal{C}_{\mathfrak{a},\mathfrak{b}}$ is non-empty, and 3.3.3, which shows the discreteness of the set $\mathcal{C}_{\mathfrak{a},\mathfrak{b}}$.

We record here a counting lemma, which will come in handy later on.

Lemma 3.2.8. *Let $\mathfrak{a}$ be a cusp for Γ, $z \in \mathbb{H}$ and $Y > 0$. We have*

$$\#\left\{\gamma \in \hat{\Gamma}_{\mathfrak{a}} \backslash \hat{\Gamma} \;\middle|\; \mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z) > Y \text{ and } \hat{\Gamma}_{\mathfrak{a}}\gamma \neq \hat{\Gamma}_{\mathfrak{a}}\right\} \leq \frac{10}{c_{\mathfrak{a}} Y}.$$

Proof. This is [Iwa02, Lemma 2.10]. ☐

3.3 KLOOSTERMAN SUMS

The classical Kloosterman sum, that appeared in Kloosterman's work [Klo26] is given by

$$S(m,n;c) = \sideset{}{'}\sum_{d \bmod (c)} e\left(\frac{md + n\bar{d}}{c}\right), \quad m,n \in \mathbb{Z}, c \in \mathbb{N}. \tag{3.10}$$

They appear as the Fourier coefficients of Poincaré series for the modular group $\mathrm{SL}_2(\mathbb{Z})$ with trivial multiplier system. In this way, we shall generalise the Kloosterman sum. For $c \in \mathcal{C}_{\mathfrak{a},\mathfrak{b}}$, we define the Kloosterman sum as follows:

$$S_{\mathfrak{a},\mathfrak{b}}^{\upsilon,\kappa}(m,n;c) = e^{-\frac{\pi i}{2}\kappa} \sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in B \backslash \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}} / B} \overline{\upsilon(\sigma_{\mathfrak{a}}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\sigma_{\mathfrak{b}}^{-1})} e\left((m+\eta_{\mathfrak{a}})\frac{a}{c} + (n+\eta_{\mathfrak{b}})\frac{d}{c}\right)$$
$$\times \sigma_{\kappa}(\sigma_{\mathfrak{a}}, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\sigma_{\mathfrak{b}}^{-1})\sigma_{\kappa}(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \sigma_{\mathfrak{b}}^{-1}). \tag{3.11}$$

Remark 3.3.1. Often in the literature, the Kloosterman sums are defined without the extra factor $e^{-\frac{\pi i}{2}\kappa}$. However, with this normalisation the Kloosterman sum $S_{\mathfrak{a},\mathfrak{a}}^{\upsilon,\kappa}(m,m;c)$ is real (see Proposition 3.3.2). It is rather unfortunate that with this normalisation the Kloosterman sum is no longer well-defined for $\kappa \bmod (2)$, but rather $\kappa \bmod (4)$, hence we carry the κ around in our notation for clarity.

The Kloosterman sums are well-defined. This will follow from Proposition 3.4.5, however it is a good exercise in the σ_κ-relations 3.2.4 to prove it directly. Let us first prove that we may replace $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} a+c & b+d \\ c & d \end{smallmatrix}\right)$. We have

$$\begin{aligned}\overline{\upsilon(\sigma_{\mathfrak{a}}\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\sigma_{\mathfrak{b}}^{-1})} &= \overline{\upsilon(\sigma_{\mathfrak{a}}\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\sigma_{\mathfrak{a}}^{-1})\upsilon(\sigma_{\mathfrak{a}}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\sigma_{\mathfrak{b}}^{-1})\sigma_\kappa(\sigma_{\mathfrak{a}}\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\sigma_{\mathfrak{a}}^{-1}, \sigma_{\mathfrak{a}}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\sigma_{\mathfrak{b}}^{-1})} \\ &= e(-\eta_{\mathfrak{a}})\overline{\upsilon(\sigma_{\mathfrak{a}}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\sigma_{\mathfrak{b}}^{-1})\sigma_\kappa(\sigma_{\mathfrak{a}}\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\sigma_{\mathfrak{a}}^{-1}, \sigma_{\mathfrak{a}}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\sigma_{\mathfrak{b}}^{-1})}\end{aligned}$$

from the definition of a multiplier system and the definition of the cusp parameter. Furthermore, we have

$$e\left((m+\eta_{\mathfrak{a}})\frac{a+c}{c} + (n+\eta_{\mathfrak{b}})\frac{d}{c}\right) = e(\eta_{\mathfrak{a}})e\left((m+\eta_{\mathfrak{a}})\frac{a}{c} + (n+\eta_{\mathfrak{b}})\frac{d}{c}\right).$$

By using (3.2) and (3.6), we get

$$\begin{aligned}\overline{\sigma_\kappa(\sigma_{\mathfrak{a}}\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\sigma_{\mathfrak{a}}^{-1}, \sigma_{\mathfrak{a}}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\sigma_{\mathfrak{b}}^{-1})} &= \frac{\sigma_\kappa(\sigma_{\mathfrak{a}}, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\sigma_{\mathfrak{b}}^{-1})}{\sigma_\kappa(\sigma_{\mathfrak{a}}\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\sigma_{\mathfrak{a}}^{-1}, \sigma_{\mathfrak{a}})}\overline{\sigma_\kappa(\sigma_{\mathfrak{a}}\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\sigma_{\mathfrak{b}}^{-1})} \\ &= \sigma_\kappa(\sigma_{\mathfrak{a}}, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\sigma_{\mathfrak{b}}^{-1})\overline{\sigma_\kappa(\sigma_{\mathfrak{a}}\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\sigma_{\mathfrak{b}}^{-1})}.\end{aligned}$$

By using (3.2) again as well as (3.3), we find

$$\begin{aligned}\sigma_\kappa(\sigma_{\mathfrak{a}}, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\sigma_{\mathfrak{b}}^{-1}) &= \frac{\sigma_\kappa(\sigma_{\mathfrak{a}}\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\sigma_{\mathfrak{b}}^{-1})\sigma_\kappa(\sigma_{\mathfrak{a}}, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right))}{\sigma_\kappa(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\sigma_{\mathfrak{b}}^{-1})} \\ &= \sigma_\kappa(\sigma_{\mathfrak{a}}\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\sigma_{\mathfrak{b}}^{-1}).\end{aligned}$$

Finally, we find by means of (3.3), that

$$\sigma_\kappa(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \sigma_{\mathfrak{b}}^{-1}) = \sigma_\kappa(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \sigma_{\mathfrak{b}}^{-1}).$$

The combination of all of these equations shows that the left quotient is well-defined. To show that the right quotient is well-defined we need to show that we can replace $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & a+b \\ c & c+d \end{smallmatrix}\right)$. Noting that

$$\sigma_\kappa(\sigma_{\mathfrak{a}}, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\sigma_{\mathfrak{b}}^{-1})\sigma_\kappa(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \sigma_{\mathfrak{b}}^{-1}) = \sigma_\kappa(\sigma_{\mathfrak{a}}, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right))\sigma_\kappa(\sigma_{\mathfrak{a}}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \sigma_{\mathfrak{b}}^{-1}),$$

we find that the proof will be pretty much identical, thus we omit the computation.

The next proposition shows that the Kloosterman sums admit some symmetries. In particular, they show that in the case when the two cusps $\mathfrak{a}, \mathfrak{b}$, and m, n are equal, respectively, the Kloosterman sum is real.

Proposition 3.3.2. *We have*

$$\overline{S_{\mathfrak{a},\mathfrak{b}}^{\upsilon,\kappa}(m,n;c)} = S_{\mathfrak{a},\mathfrak{b}}^{\overline{\upsilon},-\kappa}(-m-\delta_{\mathfrak{a}}^{ns}, -n-\delta_{\mathfrak{b}}^{ns}; c) = S_{\mathfrak{b},\mathfrak{a}}^{\upsilon,\kappa}(n,m;c)$$

Proof. The author believes there should be a short conceptual prove of this fact[a]. However, the author was not able to find one and therefore an ad hoc proof is given.

The first equality clearly holds by simply writing out the definitions. For this endeavour, one should recall that $\overline{\upsilon}$ is a multiplier system of weight $-\kappa$. The second equality requires a lot more work. We are going to make use of the involution in Lemma 3.2.7. For this matter, we need a few identities. The first one follows from the definition of a multiplier system

$$\begin{aligned}\upsilon(\sigma_{\mathfrak{a}}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\sigma_{\mathfrak{b}}^{-1}) =& \upsilon(-I)\overline{\upsilon(\sigma_{\mathfrak{b}}\left(\begin{smallmatrix} -d & b \\ c & -a \end{smallmatrix}\right)\sigma_{\mathfrak{a}}^{-1})\sigma_{\kappa}(\sigma_{\mathfrak{a}}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\sigma_{\mathfrak{b}}^{-1},\sigma_{\mathfrak{b}}\left(\begin{smallmatrix} -d & b \\ c & -a \end{smallmatrix}\right)\sigma_{\mathfrak{a}}^{-1})} \\ =& e^{-\pi i\kappa}\overline{\upsilon(\sigma_{\mathfrak{b}}\left(\begin{smallmatrix} -d & b \\ c & -a \end{smallmatrix}\right)\sigma_{\mathfrak{a}}^{-1})\sigma_{\kappa}(\sigma_{\mathfrak{a}}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\sigma_{\mathfrak{b}}^{-1},\sigma_{\mathfrak{b}}\left(\begin{smallmatrix} -d & b \\ c & -a \end{smallmatrix}\right)\sigma_{\mathfrak{a}}^{-1})}.\end{aligned}$$

The second, third, and fourth are just (3.2)

$$\begin{aligned}\sigma_{-\kappa}(\sigma_{\mathfrak{a}},\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\sigma_{\mathfrak{b}}^{-1}) =& \sigma_{\kappa}(\sigma_{\mathfrak{a}}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\sigma_{\mathfrak{b}}^{-1},\sigma_{\mathfrak{b}}\left(\begin{smallmatrix} -d & b \\ c & -a \end{smallmatrix}\right)\sigma_{\mathfrak{a}}^{-1})\sigma_{-\kappa}(\sigma_{\mathfrak{a}},-\sigma_{\mathfrak{a}}^{-1}) \\ & \cdot\sigma_{-\kappa}(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\sigma_{\mathfrak{b}}^{-1},\sigma_{\mathfrak{b}}\left(\begin{smallmatrix} -d & b \\ c & -a \end{smallmatrix}\right)\sigma_{\mathfrak{a}}^{-1}), \\ \sigma_{-\kappa}(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\sigma_{\mathfrak{b}}^{-1},\sigma_{\mathfrak{b}}\left(\begin{smallmatrix} -d & b \\ c & -a \end{smallmatrix}\right)\sigma_{\mathfrak{a}}^{-1}) =& \sigma_{\kappa}(\sigma_{\mathfrak{b}},\left(\begin{smallmatrix} -d & b \\ c & -a \end{smallmatrix}\right)\sigma_{\mathfrak{a}}^{-1})\sigma_{-\kappa}(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right),\left(\begin{smallmatrix} -d & b \\ c & -a \end{smallmatrix}\right)\sigma_{\mathfrak{a}}^{-1}) \\ & \cdot\sigma_{-\kappa}(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\sigma_{\mathfrak{b}}^{-1},\sigma_{\mathfrak{b}}), \\ \sigma_{-\kappa}(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right),\left(\begin{smallmatrix} -d & b \\ c & -a \end{smallmatrix}\right)\sigma_{\mathfrak{a}}^{-1}) =& \sigma_{\kappa}(\left(\begin{smallmatrix} -d & b \\ c & -a \end{smallmatrix}\right),\sigma_{\mathfrak{a}}^{-1})\sigma_{-\kappa}(-I,\sigma_{\mathfrak{a}}^{-1})\sigma_{-\kappa}(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right),\left(\begin{smallmatrix} -d & b \\ c & -a \end{smallmatrix}\right)).\end{aligned}$$

The fifth follows from (3.2), (3.3) and the choice of $\sigma_{\mathfrak{a}}$

$$\sigma_{-\kappa}(\sigma_{\mathfrak{a}},-\sigma_{\mathfrak{a}}^{-1}) = \sigma_{\kappa}(\sigma_{\mathfrak{a}}^{-1},-I)\sigma_{-\kappa}(I,-I)\sigma_{-\kappa}(\sigma_{\mathfrak{a}},\sigma_{\mathfrak{a}}^{-1}) = \sigma_{\kappa}(\sigma_{\mathfrak{a}}^{-1},-I).$$

The sixth is (3.2), (3.3) as well as the choice of $\sigma_{\mathfrak{b}}$

$$\sigma_{-\kappa}(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right),\sigma_{\mathfrak{b}}^{-1})\sigma_{-\kappa}(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\sigma_{\mathfrak{b}}^{-1},\sigma_{\mathfrak{b}}) = \sigma_{-\kappa}(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right),I)\sigma_{-\kappa}(\sigma_{\mathfrak{b}}^{-1},\sigma_{\mathfrak{b}}) = 1.$$

The seventh is writing out the definitions

$$\sigma_{\kappa}(\sigma_{\mathfrak{a}}^{-1},-I)\sigma_{-\kappa}(-I,\sigma_{\mathfrak{a}}^{-1}) = \frac{j(\sigma_{\mathfrak{a}}^{-1},z)^{\kappa}j(-I,z)^{\kappa}}{j(-\sigma_{\mathfrak{a}}^{-1},z)^{\kappa}}\cdot\frac{j(-\sigma_{\mathfrak{a}}^{-1},z)^{\kappa}}{j(-I,\sigma_{\mathfrak{a}}^{-1}z)^{\kappa}j(\sigma_{\mathfrak{a}}^{-1},z)^{\kappa}} = 1.$$

And finally, the last one follows from (3.2), (3.3) and (3.9)

$$\begin{aligned}\sigma_{-\kappa}(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right),\left(\begin{smallmatrix} -d & b \\ c & -a \end{smallmatrix}\right)) =& \sigma_{\kappa}(\left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right),-I)\sigma_{-\kappa}(I,-I)\sigma_{-\kappa}(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right),\left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)) \\ =& \sigma_{\kappa}(\left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right),-I) = 1.\end{aligned}$$

a A long conceptual proof can be given through the evaluation of the inner product in Proposition 3.6.7 in two ways and establishing enough analytic freedom.

Combining all of them shows

$$\upsilon(\sigma_{\mathfrak{a}} \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \sigma_{\mathfrak{b}}^{-1})\sigma_{-\kappa}(\sigma_{\mathfrak{a}}, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \sigma_{\mathfrak{b}}^{-1})\sigma_{-\kappa}(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \sigma_{\mathfrak{b}}^{-1})$$
$$= e^{-\pi i\kappa}\overline{\upsilon(\sigma_{\mathfrak{b}} \left(\begin{smallmatrix} -d & b \\ c & -a \end{smallmatrix}\right) \sigma_{\mathfrak{a}}^{-1})}\sigma_{\kappa}(\sigma_{\mathfrak{b}}, \left(\begin{smallmatrix} -d & b \\ c & -a \end{smallmatrix}\right) \sigma_{\mathfrak{a}}^{-1})\sigma_{\kappa}(\left(\begin{smallmatrix} -d & b \\ c & -a \end{smallmatrix}\right), \sigma_{\mathfrak{a}}^{-1}).$$

The only thing left to note is

$$e\left((-m-\delta_{\mathfrak{a}}^s+\eta_{\mathfrak{a}}^{\overline{\upsilon}})\frac{a}{c}+(-n-\delta_{\mathfrak{b}}^s+\eta_{\mathfrak{b}}^{\overline{\upsilon}})\frac{d}{c}\right) = e\left((n+\eta_{\mathfrak{b}}^{\upsilon})\frac{-d}{c}+(m+\eta_{\mathfrak{a}}^{\upsilon})\frac{-a}{c}\right).$$

□

We have the following trivial bounds for the Kloosterman sum.

Proposition 3.3.3. *For any $c \in \mathcal{C}_{\mathfrak{a},\mathfrak{b}}$, we have*

$$|S_{\mathfrak{a},\mathfrak{b}}^{\upsilon,\kappa}(m,n;c)| \leq \max\{c_{\mathfrak{a}}, c_{\mathfrak{b}}\}^{-1}c^2 \tag{3.12}$$

and

$$\sum_{\substack{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}} \\ c\leq X}} \frac{1}{c}|S_{\mathfrak{a},\mathfrak{b}}^{\upsilon,\kappa}(m,n;c)| \leq \max\{c_{\mathfrak{a}}, c_{\mathfrak{b}}\}^{-1}X. \tag{3.13}$$

Proof. See [Iwa02, Proposition 2.8, Corollary 2.9]. □

From this, it follows that the Kloosterman zeta function, which we shall define as

$$\mathcal{Z}_{\mathfrak{a},\mathfrak{b}}^{\upsilon,\kappa}(m,n;s) = \sum_{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}} \frac{S_{\mathfrak{a},\mathfrak{b}}^{\upsilon,\kappa}(m,n;c)}{c^{2s}}, \tag{3.14}$$

converges locally absolutely uniformly to a holomorphic function in the half-plane $\mathrm{Re}(s) > 1$. In fact, $\mathcal{Z}_{\mathfrak{a},\mathfrak{b}}^{\upsilon,\kappa}(m,n;s)$ extends to a meromorphic function in the half-plane $\mathrm{Re}(s) > \frac{1}{2}$. We refer to [Sel65].

3.4 MAASS FORMS

On the set of functions $f : \mathbb{H} \to \mathbb{C}$, we may define the slash operator $|_\kappa\gamma$ for every matrix $\gamma \in \mathrm{SL}_2(\mathbb{R})$. The operator is defined as follows

$$(f|_\kappa\gamma)(z) = \left(\frac{j(\gamma,z)}{|j(\gamma,z)|}\right)^{-\kappa} f(\gamma z)$$

and it satisfies

$$f|_\kappa\gamma\tau = \sigma_\kappa(\gamma,\tau)\cdot(f|_\kappa\gamma)|_\kappa\tau, \quad \forall\gamma,\tau\in\mathrm{SL}_2(\mathbb{R}).$$

Definition 3.4.1. Let Γ be a Fuchsian group of the first kind and υ a multiplier system of weight κ with respect to Γ. A function $f : \mathbb{H} \to \mathbb{C}$ is called *modular* with respect to υ (and Γ) if it satisfies

$$f|_\kappa \gamma = \upsilon(\gamma) f, \quad \forall \gamma \in \Gamma.$$

The set of all such functions is denoted by $\mathcal{F}_\kappa(\Gamma, \upsilon)$.

On the space $C^\infty(\mathbb{H}, \mathbb{C})$, we may also define the Laplace–Beltrami operator Δ_κ (of weight κ), which is given as

$$\Delta_\kappa = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - i\kappa y \frac{\partial}{\partial x}.$$

The operator does not agree with the usual Laplace–Beltrami operator Δ on $\mathbb{H}$, which is given by Δ_0. However, Δ_κ is linearly related to the Laplace–Beltrami operator of the universal cover $\widetilde{\mathrm{SL}_2}(\mathbb{R})$ of $\mathrm{SL}_2(\mathbb{R})$ when restricted to a certain subspace of $C^\infty(\widetilde{\mathrm{SL}_2}(\mathbb{R}), \mathbb{C})$. This justifies its name. The reader may wish to consult with [Roe66, Chapter 4] to find the details of this connection.

Lemma 3.4.1. *The Laplace–Beltrami operator Δ_κ commutes with all the slash operators $|_\kappa \gamma$.*

Proof. We have

$$\Delta_\kappa = -(z - \bar{z})^2 \frac{\partial^2}{\partial z \partial \bar{z}} - \frac{\kappa}{2}(z - \bar{z}) \left(\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}} \right),$$

where

$$\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

Before proceeding, the reader may wish to recall the chain rules

$$\frac{\partial}{\partial z}(f \circ g) = (f_z \circ g) g_z + (f_{\bar{z}} \circ g) \bar{g}_z \text{ and } \frac{\partial}{\partial \bar{z}}(f \circ g) = (f_z \circ g) g_{\bar{z}} + (f_{\bar{z}} \circ g) \overline{g_z},$$

where f_z respectively $f_{\bar{z}}$ denotes the partial derivative with respect to z respectively $\bar{z}$. Now, let $f \in C^2(\mathbb{H}, \mathbb{C})$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$. Then, we have that $((\Delta_\kappa f)|_\kappa \gamma)(z)$ equals

$$- \; j(\gamma, z)^{-\frac{\kappa}{2}} j(\gamma, \bar{z})^{\frac{\kappa}{2}} \left[\frac{(z - \bar{z})^2}{j(\gamma, z)^2 j(\gamma, \bar{z})^2} f_{\bar{z}z}(\gamma z) + \frac{\kappa}{2} \frac{z - \bar{z}}{j(\gamma, z) j(\gamma, \bar{z})} (f_z(\gamma z) + f_{\bar{z}}(\gamma z)) \right].$$

Next, we have

$$\frac{\partial}{\partial z}(f|_\kappa \gamma)(z) = -c \frac{\kappa}{2} j(\gamma, z)^{-\frac{\kappa}{2} - 1} j(\gamma, \bar{z})^{\frac{\kappa}{2}} f(\gamma z) + j(\gamma, z)^{-\frac{\kappa}{2} - 2} j(\gamma, \bar{z})^{\frac{\kappa}{2}} f_z(\gamma z),$$
$$\frac{\partial}{\partial \bar{z}}(f|_\kappa \gamma)(z) = c \frac{\kappa}{2} j(\gamma, z)^{-\frac{\kappa}{2}} j(\gamma, \bar{z})^{\frac{\kappa}{2} - 1} f(\gamma z) + j(\gamma, z)^{-\frac{\kappa}{2}} j(\gamma, \bar{z})^{\frac{\kappa}{2} - 2} f_{\bar{z}}(\gamma z),$$

and furthermore

$$\begin{aligned}\frac{\partial^2}{\partial z \partial \bar{z}}(f|_\kappa\gamma)(z) = & -c^2\frac{\kappa^2}{4}j(\gamma,z)^{-\frac{\kappa}{2}-1}j(\gamma,\bar{z})^{\frac{\kappa}{2}-1}f(\gamma z) + c\frac{\kappa}{2}j(\gamma,z)^{-\frac{\kappa}{2}-2}j(\gamma,\bar{z})^{\frac{\kappa}{2}-1}f_z(\gamma z) \\ & - c\frac{\kappa}{2}j(\gamma,z)^{-\frac{\kappa}{2}-1}j(\gamma,\bar{z})^{\frac{\kappa}{2}-2}f_{\bar{z}}(\gamma z) + j(\gamma,z)^{-\frac{\kappa}{2}-2}j(\gamma,\bar{z})^{\frac{\kappa}{2}-2}f_{\bar{z}z}(\gamma z).\end{aligned}$$

Hence,

$$\begin{aligned}\Delta_\kappa(f|_\kappa\gamma)(z) = & -(z-\bar{z})^2 j(\gamma,z)^{-\frac{\kappa}{2}-2}j(\gamma,\bar{z})^{\frac{\kappa}{2}-2}f_{\bar{z}z}(\gamma z) \\ & - \frac{\kappa}{2}(z-\bar{z})j(\gamma,z)^{-\frac{\kappa}{2}-2}j(\gamma,\bar{z})^{\frac{\kappa}{2}-1}\left(j(\gamma,\bar{z}) + (z-\bar{z})c\right)f_z(\gamma z) \\ & - \frac{\kappa}{2}(z-\bar{z})j(\gamma,z)^{-\frac{\kappa}{2}-1}j(\gamma,\bar{z})^{\frac{\kappa}{2}-2}\left(j(\gamma,z) - (z-\bar{z})c\right)f_{\bar{z}}(\gamma z) \\ & + c\frac{\kappa^2}{4}(z-\bar{z})j(\gamma,z)^{-\frac{\kappa}{2}-1}j(\gamma,\bar{z})^{\frac{\kappa}{2}-1}\left(c(z-\bar{z}) + j(\gamma,\bar{z}) - j(\gamma,z)\right)f(\gamma z) \\ = & ((\Delta_\kappa f)|_\kappa\gamma)(z).\end{aligned}$$

□

Therefore, Δ_κ operates on $\mathcal{A}_\kappa^\infty(\Gamma, \upsilon) = C^\infty(\mathbb{H}, \mathbb{C}) \cap \mathcal{F}_\kappa(\Gamma, \upsilon)$ and we are now able to define what a Maass form is.

Definition 3.4.2. A *Maass form* with respect to the Fuchsian group Γ, multiplier system υ of weight κ is a non-zero function $f \in \mathcal{A}_\kappa^\infty(\Gamma, \upsilon)$, which satisfies the following properties

1. f is real analytic in x and y simultaneously,
2. $\exists \lambda_f \in \mathbb{C} : -\Delta_\kappa f = \lambda_f f$,
3. For all cusps $\mathfrak{a}$ of Γ, we have $(f|_\kappa\sigma_\mathfrak{a})(z) = y^{O(1)}$ as $y \to \infty$ uniformly in x.

λ_f is called the eigenvalue of f (with respect to Δ_κ). The span of all Maass forms with respect to Γ and υ of weight κ and eigenvalue λ is denoted by $\mathcal{A}_\kappa^\infty(\Gamma, \upsilon, \lambda)$.

We can expand a Maass form f at a cusp $\mathfrak{a}$ of Γ as a series. For this matter, we let t_f satisfy the equation $\lambda_f = \frac{1}{4} + t_f^2$ and let us denote with $W_{k,m}(z)$ the Whittaker function, which is defined in (A.1). If λ_f happens to be real, then we make the convention that $t_f \in \mathbb{R}_0^+ \cup i\mathbb{R}_0^+$. The expansion takes the following shape [Roe66, Chapter 2]:

$$(f|_\kappa\sigma_\mathfrak{a})(z) = \sum_{m\in\mathbb{Z}} c_f(\mathfrak{a}, m; y)e((m+\eta_\mathfrak{a})x), \tag{3.15}$$

where

$$c_f(\mathfrak{a}, m; y) = \begin{cases} \rho_f(\mathfrak{a}, m)W_{\operatorname{sign}(m+\eta_\mathfrak{a})\frac{\kappa}{2}, it_f}(4\pi|m+\eta_\mathfrak{a}|y), & m+\eta_\mathfrak{a} \neq 0, \\ \rho_f(\mathfrak{a}, 0)y^{\frac{1}{2}+it_f} + \rho_f'(\mathfrak{a}, 0)y^{\frac{1}{2}-it_f}, & m = \eta_\mathfrak{a} = 0 \neq t_f, \\ \rho_f(\mathfrak{a}, 0)y^{\frac{1}{2}} + \rho_f'(\mathfrak{a}, 0)y^{\frac{1}{2}}\log(y), & m = \eta_\mathfrak{a} = t_f = 0. \end{cases}$$

This expansion converges absolutely uniformly for $y \geq y_0$ and we shall refer to it as the Fourier expansion. We say a Maass form is *cuspidal* if the coefficients $\rho_f(\mathfrak{a}, 0)$ and $\rho'_f(\mathfrak{a}, 0)$ vanish for all singular cusps.

Besides the Laplace–Beltrami operator Δ_κ, there are also two further operators which increase, respectively decrease, the weight of a function. They are defined as follows:

$$K_\kappa = iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{\kappa}{2} = (z - \overline{z})\frac{\partial}{\partial z} + \frac{\kappa}{2},$$
$$\Lambda_\kappa = iy\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + \frac{\kappa}{2} = (z - \overline{z})\frac{\partial}{\partial \overline{z}} + \frac{\kappa}{2}.$$

We have the following lemma.

Lemma 3.4.2. *Let $f \in C^\infty(\mathbb{H}, \mathbb{C})$ and $\gamma \in \mathrm{SL}_2(\mathbb{R})$. Then, we have*

$$\begin{aligned}
(K_\kappa f)|_{\kappa+2}\gamma &= K_\kappa(f|_\kappa\gamma), \\
(\Lambda_\kappa f)|_{\kappa-2}\gamma &= \Lambda_\kappa(f|_\kappa\gamma), \\
-\Delta_\kappa &= \Lambda_{\kappa+2}K_\kappa - \tfrac{\kappa}{2}(1 + \tfrac{\kappa}{2}), \\
-\Delta_\kappa &= K_{\kappa-2}\Lambda_\kappa + \tfrac{\kappa}{2}(1 - \tfrac{\kappa}{2}), \\
\Delta_{\kappa+2}K_\kappa &= K_\kappa\Delta_\kappa, \\
\Delta_{\kappa-2}\Lambda_\kappa &= \Lambda_\kappa\Delta_\kappa.
\end{aligned}$$

Proof. See [Roe66, pp. 305-306] □

This shows that K_κ maps $\mathcal{A}^\infty_\kappa(\Gamma, \upsilon, \lambda)$ to $\mathcal{A}^\infty_{\kappa+2}(\Gamma, \upsilon, \lambda)$ and Λ_κ maps $\mathcal{A}^\infty_\kappa(\Gamma, \upsilon, \lambda)$ to $\mathcal{A}^\infty_{\kappa-2}(\Gamma, \upsilon, \lambda)$, which are bijections as long as $\lambda \neq -\frac{\kappa}{2}(1 + \frac{\kappa}{2})$, respectively $\lambda \neq \frac{\kappa}{2}(1 - \frac{\kappa}{2})$. We shall classify at a later stage what happens at these special eigenvalues (see Lemma 3.5.2). We shall examine what happens to the Fourier coefficient of a Maass form under the increase and decrease operator.

Lemma 3.4.3. *Let $f \in \mathcal{A}^\infty_\kappa(\Gamma, \upsilon, \lambda)$. Then, we have for the increase operator K_κ:*

$$\rho_{K_\kappa f}(\mathfrak{a}, m) = \rho_f(\mathfrak{a}, m) \times \begin{cases} -1, & m + \eta_\mathfrak{a} > 0, \\ \left(t_f^2 + (\frac{\kappa}{2} + \frac{1}{2})^2\right), & m + \eta_\mathfrak{a} < 0, \end{cases}$$

if $m = \eta_\mathfrak{a} = 0 \neq t_f$, then we have

$$\begin{aligned}
\rho_{K_\kappa f}(\mathfrak{a}, 0) &= (\tfrac{1}{2} + \tfrac{\kappa}{2} + it_f)\rho_f(\mathfrak{a}, 0), \\
\rho'_{K_\kappa f}(\mathfrak{a}, 0) &= (\tfrac{1}{2} + \tfrac{\kappa}{2} - it_f)\rho'_f(\mathfrak{a}, 0),
\end{aligned}$$

and finally if $m = \eta_{\mathfrak{a}} = t_f = 0$, then

$$\rho_{K_\kappa f}(\mathfrak{a}, 0) = (\tfrac{1}{2} + \tfrac{\kappa}{2})\rho_f(\mathfrak{a}, 0) + \rho'_f(\mathfrak{a}, 0),$$
$$\rho'_{K_\kappa f}(\mathfrak{a}, 0) = (\tfrac{1}{2} + \tfrac{\kappa}{2})\rho'_f(\mathfrak{a}, 0).$$

For the decrease operator Λ_κ, we have similarly:

$$\rho_{\Lambda_\kappa f}(\mathfrak{a}, m) = \rho_f(\mathfrak{a}, m) \times \begin{cases} -\left(t_f^2 + (\frac{\kappa}{2} - \frac{1}{2})^2\right), & m + \eta_{\mathfrak{a}} > 0, \\ 1, & m + \eta_{\mathfrak{a}} < 0, \end{cases}$$

if $m = \eta_{\mathfrak{a}} = 0 \neq t_f$, then we have

$$\rho_{\Lambda_\kappa f}(\mathfrak{a}, 0) = (-\tfrac{1}{2} + \tfrac{\kappa}{2} - it_f)\rho_f(\mathfrak{a}, 0),$$
$$\rho'_{\Lambda_\kappa f}(\mathfrak{a}, 0) = (-\tfrac{1}{2} + \tfrac{\kappa}{2} + it_f)\rho'_f(\mathfrak{a}, 0),$$

and finally if $m = \eta_{\mathfrak{a}} = t_f = 0$, then

$$\rho_{\Lambda_\kappa f}(\mathfrak{a}, 0) = (-\tfrac{1}{2} + \tfrac{\kappa}{2})\rho_f(\mathfrak{a}, 0) - \rho'_f(\mathfrak{a}, 0),$$
$$\rho'_{\Lambda_\kappa f}(\mathfrak{a}, 0) = (-\tfrac{1}{2} + \tfrac{\kappa}{2})\rho'_f(\mathfrak{a}, 0).$$

Proof. This is recorded in [AA18, Eq. (2.16)] or easily verified by means of the relation amongst the Whittaker functions (A.3) and (A.2). □

There is a continuum of important examples of Maass forms[b], namely, the Eisenstein series attached to a singular cusp. They will arise as a special case of the more general Poincaré series, which we shall define here.

Definition 3.4.3. Let v be a multiplier system of weight κ for Γ, $\mathfrak{a}$ a cusp of Γ, and $m \in \mathbb{Z}$ be an integer with $m + \eta_{\mathfrak{a}} \geq 0$. Then, we define the *$m$-th Poincaré series* attached to the cusp $\mathfrak{a}$ with respect to the multiplier system v as

$$\mathcal{U}^{v,\kappa}_{\mathfrak{a},m}(z, s) = \sum_{\gamma \in \hat{\Gamma}_{\mathfrak{a}} \backslash \hat{\Gamma}} \overline{v(\gamma)\sigma_\kappa(\sigma_{\mathfrak{a}}^{-1}, \gamma)} \operatorname{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^s e((m + \eta_{\mathfrak{a}})\sigma_{\mathfrak{a}}^{-1}\gamma z) \left(\frac{j(\sigma_{\mathfrak{a}}^{-1}\gamma, z)}{|j(\sigma_{\mathfrak{a}}^{-1}\gamma, z)|}\right)^{-\kappa},$$

where $z \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 1$.

The Poincaré series exhibit a defining property which we shall show later in Proposition 3.6.2. The Eisenstein series now arise in the special case when the cusp $\mathfrak{a}$ is singular.

b Pun intended.

Definition 3.4.4. Let υ be a multiplier system of weight κ for Γ and $\mathfrak{a}$ a singular cusp of Γ. Then, the 0-th Poincaré series attached to the cusp $\mathfrak{a}$ with respect to the multiplier system υ is called the *Eisenstein series* attached to the cusp $\mathfrak{a}$ with respect to the multiplier system υ and write

$$\mathcal{E}_{\mathfrak{a}}^{\upsilon,\kappa}(z,s) = \mathcal{U}_{\mathfrak{a},0}^{\upsilon,\kappa}(z,s).$$

Proposition 3.4.4. *The Poincaré series are well-defined and converge absolutely locally uniformly on $\mathbb{H} \times \{s \in \mathbb{C} | \operatorname{Re}(s) > 1\}$. They are furthermore modular with respect to υ and Γ.*

Proof. Let $\tau \in \Gamma_{\mathfrak{a}} := \sigma_{\mathfrak{a}} B \sigma_{\mathfrak{a}}^{-1}$, say $\tau = \sigma_{\mathfrak{a}} \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) \sigma_{\mathfrak{a}}^{-1}$ with $n \in \mathbb{Z}$. Then, we have

$$\begin{aligned} \operatorname{Im}(\sigma_{\mathfrak{a}}^{-1}\tau\gamma z) &= \operatorname{Im}(\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) \sigma_{\mathfrak{a}}^{-1}\gamma z) = \operatorname{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z), \\ j(\sigma_{\mathfrak{a}}^{-1}\tau\gamma, z) &= j(\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) \sigma_{\mathfrak{a}}^{-1}\gamma, z) = j(\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right), \sigma_{\mathfrak{a}}^{-1}\gamma z) j(\sigma_{\mathfrak{a}}^{-1}\gamma, z) = j(\sigma_{\mathfrak{a}}^{-1}\gamma, z), \\ e((m+\eta_{\mathfrak{a}})\sigma_{\mathfrak{a}}^{-1}\tau\gamma z) &= e((m+\eta_{\mathfrak{a}})(\sigma_{\mathfrak{a}}^{-1}\gamma z + n)) = e((m+\eta_{\mathfrak{a}})\sigma_{\mathfrak{a}}^{-1}\gamma z) e(n\eta_{\mathfrak{a}}), \\ \overline{\upsilon(\tau\gamma)} &= \overline{\upsilon(\tau)\upsilon(\gamma)\sigma_\kappa(\tau,\gamma)} = e(-n\eta_{\mathfrak{a}})\overline{\upsilon(\gamma)\sigma_\kappa(\tau,\gamma)}. \end{aligned}$$

With the help of (3.2),(3.3),(3.4), we find

$$\overline{\sigma_\kappa(\tau,\gamma)\sigma_\kappa(\sigma_{\mathfrak{a}}^{-1},\tau\gamma)} = \overline{\sigma_\kappa(\sigma_{\mathfrak{a}}^{-1},\tau)\sigma_\kappa(\sigma_{\mathfrak{a}}^{-1}\tau,\gamma)} = \overline{\sigma_\kappa(\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)\sigma_{\mathfrak{a}}^{-1},\gamma)} = \overline{\sigma_\kappa(\sigma_{\mathfrak{a}}^{-1},\gamma)}.$$

This shows that the sum is well-defined modulo $\Gamma_{\mathfrak{a}}$. In order to show that the sum is also well-defined modulo ± 1, it is sufficient to show that

$$\upsilon(\gamma)\sigma_\kappa(\sigma_{\mathfrak{a}}^{-1},\gamma)\left(\frac{j(\sigma_{\mathfrak{a}}^{-1}\gamma,z)}{|j(\sigma_{\mathfrak{a}}^{-1}\gamma,z)|}\right)^\kappa = \upsilon(\gamma) j(\gamma,z)^\kappa \cdot j(\sigma_{\mathfrak{a}}^{-1},\gamma z)^\kappa \cdot \frac{1}{|j(\sigma_{\mathfrak{a}}^{-1}\gamma,z)|^\kappa}$$

remains unchanged when replacing γ with $-\gamma$. However, this is clearly the case since each factor on the right-hand side remains unchanged, thereby concluding that the sum is well-defined.

Let us now turn our attention to the uniform convergence of the series. We have

$$|\mathcal{U}_{\mathfrak{a},m}^{\upsilon,\kappa}(z,s)| \leq \sum_{\gamma \in \hat{\Gamma}_{\mathfrak{a}} \setminus \hat{\Gamma}} \operatorname{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^{\operatorname{Re}(s)} e^{-2\pi(m+\eta_{\mathfrak{a}})\operatorname{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)}.$$

Let us first consider the case $m + \eta_{\mathfrak{a}} > 0$. The function $x^\alpha e^{-\beta x}$ $(x, \alpha, \beta > 0)$ takes its maximum of $(\frac{\alpha}{e\beta})^\alpha$ at $x = \frac{\alpha}{\beta}$. Thus, the sum stemming from the $\gamma \in \hat{\Gamma}_{\mathfrak{a}} \setminus \hat{\Gamma}$ with $\operatorname{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z) > \frac{\operatorname{Re}(s)}{2\pi(m+\eta_{\mathfrak{a}})}$ is bounded by

$$\left(\frac{\operatorname{Re}(s)}{2e\pi(m+\eta_{\mathfrak{a}})}\right)^{\operatorname{Re}(s)} \left(1 + \frac{20\pi(m+\eta_{\mathfrak{a}})}{c_{\mathfrak{a}}\operatorname{Re}(s)}\right)$$

by Lemma 3.2.8. The remaining sum is bounded by

$$
\begin{aligned}
&-\int_0^{\frac{\operatorname{Re}(s)}{2\pi(m+\eta_{\mathfrak{a}})}} Y^{\operatorname{Re}(s)} e^{-2\pi(m+\eta_{\mathfrak{a}})Y} d\left(\sum_{\substack{\gamma\in\hat{\Gamma}_{\mathfrak{a}}\backslash\hat{\Gamma}\\ \frac{\operatorname{Re}(s)}{2\pi(m+\eta_{\mathfrak{a}})}\geq\operatorname{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)>Y}} 1\right)\\
&=\int_0^{\frac{\operatorname{Re}(s)}{2\pi(m+\eta_{\mathfrak{a}})}} \left(\sum_{\substack{\gamma\in\hat{\Gamma}_{\mathfrak{a}}\backslash\hat{\Gamma}\\ \frac{\operatorname{Re}(s)}{2\pi(m+\eta_{\mathfrak{a}})}\geq\operatorname{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)>Y}} 1\right) d\left(Y^{\operatorname{Re}(s)} e^{-2\pi(m+\eta_{\mathfrak{a}})Y}\right)\\
&\ll\int_0^{\frac{\operatorname{Re}(s)}{2\pi(m+\eta_{\mathfrak{a}})}} \left(1+\frac{10}{c_{\mathfrak{a}}Y}\right) d\left(Y^{\operatorname{Re}(s)} e^{-2\pi(m+\eta_{\mathfrak{a}})Y}\right)\\
&=\left(\frac{\operatorname{Re}(s)}{2e\pi(m+\eta_{\mathfrak{a}})}\right)^{\operatorname{Re}(s)}\left(1+\frac{20\pi(m+\eta_{\mathfrak{a}})}{c_{\mathfrak{a}}\operatorname{Re}(s)}\right)-\int_0^{\frac{\operatorname{Re}(s)}{2\pi(m+\eta_{\mathfrak{a}})}} Y^{\operatorname{Re}(s)} e^{-2\pi(m+\eta_{\mathfrak{a}})Y} d\left(1+\frac{10}{c_{\mathfrak{a}}Y}\right)\\
&\ll\left(\frac{\operatorname{Re}(s)}{2e\pi(m+\eta_{\mathfrak{a}})}\right)^{\operatorname{Re}(s)}\left(1+\frac{20\pi(m+\eta_{\mathfrak{a}})}{c_{\mathfrak{a}}\operatorname{Re}(s)}\right)+\frac{1}{c_{\mathfrak{a}}}(2\pi(m+\eta_a))^{1-\operatorname{Re}(s)}\Gamma\left(\operatorname{Re}(s)-1\right),
\end{aligned}
$$

where we have used integration by parts and Lemma 3.2.8. This shows the proclaimed convergence on the mentioned sets. The case $m+\eta_{\mathfrak{a}}=0$ is very similar. First, note that we may only consider the sum where $\hat{\Gamma}_{\mathfrak{a}}\gamma\neq\hat{\Gamma}_{\mathfrak{a}}$. In this case, we have the bound

$$
\begin{aligned}
&-\int_0^{\frac{20}{c_{\mathfrak{a}}}} Y^{\operatorname{Re}(s)} d\left(\sum_{\substack{\gamma\in\hat{\Gamma}_{\mathfrak{a}}\backslash\hat{\Gamma}-\hat{\Gamma}_{\mathfrak{a}}\\ \operatorname{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)>Y}} 1\right)\\
&=\int_0^{\frac{20}{c_{\mathfrak{a}}}} \left(\sum_{\substack{\gamma\in\hat{\Gamma}_{\mathfrak{a}}\backslash\hat{\Gamma}-\hat{\Gamma}_{\mathfrak{a}}\\ \operatorname{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)>Y}} 1\right) d\left(Y^{\operatorname{Re}(s)}\right)\\
&\ll\int_0^{\frac{20}{c_{\mathfrak{a}}}} \frac{10}{c_{\mathfrak{a}}Y}\operatorname{Re}(s)Y^{\operatorname{Re}(s)-1}dY\\
&=\frac{10}{c_{\mathfrak{a}}}\frac{\operatorname{Re}(s)}{\operatorname{Re}(s)-1}\left(\frac{20}{c_{\mathfrak{a}}}\right)^{\operatorname{Re}(s)-1},
\end{aligned}
$$

which again shows the proclaimed convergence.

We move onto the last claim. We have

$$
\mathcal{U}_{\mathfrak{a},m}^{\upsilon,\kappa}(\tau z,s)=\sum_{\gamma\in\hat{\Gamma}_{\mathfrak{a}}\backslash\hat{\Gamma}}\overline{\upsilon(\gamma)\sigma_\kappa(\sigma_{\mathfrak{a}}^{-1},\gamma)}\operatorname{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma\tau z)^s e((m+\eta_{\mathfrak{a}})\sigma_{\mathfrak{a}}^{-1}\gamma\tau z)\left(\frac{j(\sigma_{\mathfrak{a}}^{-1}\gamma,\tau z)}{|j(\sigma_{\mathfrak{a}}^{-1}\gamma,\tau z)|}\right)^{-\kappa}.
$$

Inserting the equations

$$\overline{\upsilon(\gamma)} = \overline{\upsilon(\gamma\tau)}\sigma_\kappa(\gamma,\tau)\upsilon(\tau),$$
$$\overline{\sigma_\kappa(\sigma_{\mathfrak{a}}^{-1},\gamma)} = \overline{\sigma_\kappa(\sigma_{\mathfrak{a}}^{-1},\gamma\tau)\sigma_\kappa(\gamma,\tau)}\sigma_\kappa(\sigma_{\mathfrak{a}}^{-1}\gamma,\tau),$$

which follow from the definition of a multiplier system and (3.2), respectively, as well as

$$\frac{|j(\sigma_{\mathfrak{a}}^{-1}\gamma,\tau z)|^\kappa}{j(\sigma_{\mathfrak{a}}^{-1}\gamma,\tau z)^\kappa} = \frac{|j(\sigma_{\mathfrak{a}}^{-1}\gamma\tau,z)|^\kappa}{|j(\tau,z)|^\kappa}\cdot\frac{j(\tau,z)^\kappa}{j(\sigma_{\mathfrak{a}}^{-1}\gamma\tau,z)^\kappa}\overline{\sigma_\kappa(\sigma_{\mathfrak{a}}^{-1}\gamma,\tau)},$$

shows

$$\mathcal{U}_{\mathfrak{a},m}^{\upsilon,\kappa}(\tau z,s) = \upsilon(\tau)\frac{j(\tau,z)^\kappa}{|j(\tau,z)|^\kappa}\mathcal{U}_{\mathfrak{a},m}^{\upsilon,\kappa}(z,s),$$

which is exactly what we needed to show. □

Proposition 3.4.5. *The Poincaré series admit the following Fourier expansion:*

$$\begin{aligned}(\mathcal{U}_{\mathfrak{a},m}^{\upsilon,\kappa}|_\kappa\sigma_{\mathfrak{b}})(z,s) &= \delta_{\mathfrak{a},\mathfrak{b}}y^s e((m+\eta_{\mathfrak{a}})z)\\ &+ y^s\sum_{n\in\mathbb{Z}}\left(\sum_{c\in C_{\mathfrak{a},\mathfrak{b}}}\frac{S_{\mathfrak{a},\mathfrak{b}}^{\upsilon,\kappa}(m,n;c)}{c^{2s}}B^\kappa(c,m+\eta_{\mathfrak{a}},n+\eta_{\mathfrak{b}},y,s)\right)e((n+\eta_{\mathfrak{b}})x),\end{aligned} \tag{3.16}$$

where $S_{\mathfrak{a},\mathfrak{b}}^{\upsilon,\kappa}(m,n;c)$ *is the Kloosterman sum as in* (3.11) *and*

$$B^\kappa(c,m,n,y,s) = e^{\frac{\pi i}{2}\kappa}\int_{-\infty}^{\infty} e\left(-\frac{m}{c^2(t+iy)}-nt\right)e^{-i\kappa\arg(t+iy)}\frac{dt}{(t^2+y^2)^s}. \tag{3.17}$$

Proof. We have

$$\begin{aligned}(\mathcal{U}_{\mathfrak{a},m}^{\upsilon,\kappa}|_\kappa\sigma_{\mathfrak{b}})(z,s) &= \left(\frac{j(\sigma_{\mathfrak{b}},z)}{|j(\sigma_{\mathfrak{b}},z)|}\right)^{-\kappa}\\ \times &\sum_{\gamma\in\hat\Gamma_{\mathfrak{a}}\backslash\hat\Gamma}\overline{\upsilon(\gamma)\sigma_\kappa(\sigma_{\mathfrak{a}}^{-1},\gamma)}\,\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{b}}z)^s e((m+\eta_{\mathfrak{a}})\sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{b}}z)\left(\frac{j(\sigma_{\mathfrak{a}}^{-1}\gamma,\sigma_{\mathfrak{b}}z)}{|j(\sigma_{\mathfrak{a}}^{-1}\gamma,\sigma_{\mathfrak{b}}z)|}\right)^{-\kappa}.\end{aligned}$$

We make use of the bijection $\hat\Gamma_{\mathfrak{a}}\backslash\hat\Gamma\to\hat B\backslash\sigma_{\mathfrak{a}}^{-1}\hat\Gamma\sigma_{\mathfrak{b}}$ given by $\gamma\mapsto\tau=\sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{b}}$ and find

$$\begin{aligned}(\mathcal{U}_{\mathfrak{a},m}^{\upsilon,\kappa}|_\kappa\sigma_{\mathfrak{b}})(z,s) &= \left(\frac{j(\sigma_{\mathfrak{b}},z)}{|j(\sigma_{\mathfrak{b}},z)|}\right)^{-\kappa}\\ \times \sum_{\tau\in\hat B\backslash\sigma_{\mathfrak{a}}^{-1}\hat\Gamma\sigma_{\mathfrak{b}}} &\overline{\upsilon(\sigma_{\mathfrak{a}}\tau\sigma_{\mathfrak{b}}^{-1})\sigma_\kappa(\sigma_{\mathfrak{a}}^{-1},\sigma_{\mathfrak{a}}\tau\sigma_{\mathfrak{b}}^{-1})}\,\mathrm{Im}(\tau z)^s e((m+\eta_{\mathfrak{a}})\tau z)\left(\frac{j(\tau\sigma_{\mathfrak{b}}^{-1},\sigma_{\mathfrak{b}}z)}{|j(\tau\sigma_{\mathfrak{b}}^{-1},\sigma_{\mathfrak{b}}z)|}\right)^{-\kappa}.\end{aligned}$$

We further simplify by making use of the relations

$$\left(\frac{j(\sigma_{\mathfrak{b}},z)}{|j(\sigma_{\mathfrak{b}},z)|}\right)^{-\kappa}\left(\frac{j(\tau\sigma_{\mathfrak{b}}^{-1},\sigma_{\mathfrak{b}}z)}{|j(\tau\sigma_{\mathfrak{b}}^{-1},\sigma_{\mathfrak{b}}z)|}\right)^{-\kappa} = \overline{\sigma_\kappa(\tau\sigma_{\mathfrak{b}}^{-1},\sigma_{\mathfrak{b}})}\left(\frac{j(\tau,z)}{|j(\tau,z)|}\right)^{-\kappa},$$

$$\overline{\sigma_\kappa(\tau\sigma_{\mathfrak{b}}^{-1},\sigma_{\mathfrak{b}})} = \overline{\sigma_\kappa(\tau,I)\sigma_\kappa(\sigma_{\mathfrak{b}}^{-1},\sigma_{\mathfrak{b}})}\sigma_\kappa(\tau,\sigma_{\mathfrak{b}}^{-1}) = \sigma_\kappa(\tau,\sigma_{\mathfrak{b}}^{-1}),$$

as well as

$$\overline{\sigma_\kappa(\sigma_{\mathfrak{a}}^{-1},\sigma_{\mathfrak{a}}\tau\sigma_{\mathfrak{b}}^{-1})} = \overline{\sigma_\kappa(I,\tau\sigma_{\mathfrak{b}}^{-1})\sigma_\kappa(\sigma_{\mathfrak{a}}^{-1},\sigma_{\mathfrak{a}})}\sigma_\kappa(\sigma_{\mathfrak{a}},\tau\sigma_{\mathfrak{b}}^{-1}) = \sigma_\kappa(\sigma_{\mathfrak{a}},\tau\sigma_{\mathfrak{b}}^{-1}),$$

which follow from (3.2) and (3.3). We conclude

$$\begin{aligned}&(\mathcal{U}^{\upsilon,\kappa}_{\mathfrak{a},m}|_\kappa\sigma_{\mathfrak{b}})(z,s)\\ &= \sum_{\tau\in\hat{B}\setminus\sigma_{\mathfrak{a}}^{-1}\hat{\Gamma}\sigma_{\mathfrak{b}}} \overline{\upsilon(\sigma_{\mathfrak{a}}\tau\sigma_{\mathfrak{b}}^{-1})}\sigma_\kappa(\sigma_{\mathfrak{a}},\tau\sigma_{\mathfrak{b}}^{-1})\sigma_\kappa(\tau,\sigma_{\mathfrak{b}}^{-1})\operatorname{Im}(\tau z)^s e((m+\eta_{\mathfrak{a}})\tau z)\left(\frac{j(\tau,z)}{|j(\tau,z)|}\right)^{-\kappa}.\end{aligned}$$

We are now at a state where we can make use of the double-coset representation given by Proposition 3.2.6. The contribution from $\hat{B}$, which is only present if the cusps $\mathfrak{a}$ and $\mathfrak{b}$ are equivalent, is given by

$$\delta_{\mathfrak{a},\mathfrak{b}}\overline{\upsilon(\sigma_{\mathfrak{a}}\sigma_{\mathfrak{b}}^{-1})}\sigma_\kappa(\sigma_{\mathfrak{a}},\sigma_{\mathfrak{b}}^{-1})\sigma_\kappa(I,\sigma_{\mathfrak{b}}^{-1})y^s e((m+\eta_{\mathfrak{a}})z)\left(\frac{j(I,z)}{|j(I,z)|}\right)^{-\kappa} = \delta_{\mathfrak{a},\mathfrak{b}}y^s e((m+\eta_{\mathfrak{a}})z). \tag{3.18}$$

Let $c\in C_{\mathfrak{a},\mathfrak{b}}$, $d \bmod c\mathbb{Z}$, $n\in\mathbb{Z}$ and consider the contribution from the matrix $\omega_{c,d}\left(\begin{smallmatrix}1&n\\0&1\end{smallmatrix}\right)$. We have

$$\overline{\upsilon(\sigma_{\mathfrak{a}}\omega_{c,d}\left(\begin{smallmatrix}1&n\\0&1\end{smallmatrix}\right)\sigma_{\mathfrak{b}}^{-1})} = \overline{\upsilon(\sigma_{\mathfrak{a}}\omega_{c,d}\sigma_{\mathfrak{b}}^{-1})\upsilon(\sigma_{\mathfrak{b}}\left(\begin{smallmatrix}1&n\\0&1\end{smallmatrix}\right)\sigma_{\mathfrak{b}}^{-1})\sigma_\kappa(\sigma_{\mathfrak{a}}\omega_{c,d}\sigma_{\mathfrak{b}}^{-1},\sigma_{\mathfrak{b}}\left(\begin{smallmatrix}1&n\\0&1\end{smallmatrix}\right)\sigma_{\mathfrak{b}}^{-1})}. \tag{3.19}$$

Now, we claim

$$\upsilon(\sigma_{\mathfrak{b}}\left(\begin{smallmatrix}1&n\\0&1\end{smallmatrix}\right)\sigma_{\mathfrak{b}}^{-1}) = e(\eta_{\mathfrak{b}}n). \tag{3.20}$$

This is certainly true for $n=0,1$ and therefore it suffices to prove

$$\sigma_\kappa(\sigma_{\mathfrak{b}}\left(\begin{smallmatrix}1&a\\0&1\end{smallmatrix}\right)\sigma_{\mathfrak{b}}^{-1},\sigma_{\mathfrak{b}}\left(\begin{smallmatrix}1&b\\0&1\end{smallmatrix}\right)\sigma_{\mathfrak{b}}^{-1}) = 1, \quad \forall a,b\in\mathbb{Z}.$$

Now, by (3.2), (3.5), and (3.6), we have

$$\begin{aligned}\sigma_\kappa(\sigma_{\mathfrak{b}}\left(\begin{smallmatrix}1&a\\0&1\end{smallmatrix}\right)\sigma_{\mathfrak{b}}^{-1},\sigma_{\mathfrak{b}}\left(\begin{smallmatrix}1&b\\0&1\end{smallmatrix}\right)\sigma_{\mathfrak{b}}^{-1}) &= \frac{\sigma_\kappa(\sigma_{\mathfrak{b}}\left(\begin{smallmatrix}1&a\\0&1\end{smallmatrix}\right),\left(\begin{smallmatrix}1&b\\0&1\end{smallmatrix}\right)\sigma_{\mathfrak{b}}^{-1})\sigma_\kappa(\sigma_{\mathfrak{b}}^{-1},\sigma_{\mathfrak{b}}\left(\begin{smallmatrix}1&b\\0&1\end{smallmatrix}\right)\sigma_{\mathfrak{b}}^{-1})}{\sigma_\kappa(\sigma_{\mathfrak{b}}\left(\begin{smallmatrix}1&a\\0&1\end{smallmatrix}\right),\sigma_{\mathfrak{b}}^{-1})}\\ &= \frac{\sigma_\kappa(\sigma_{\mathfrak{b}}\left(\begin{smallmatrix}1&a+b\\0&1\end{smallmatrix}\right),\sigma_{\mathfrak{b}}^{-1})}{\sigma_\kappa(\sigma_{\mathfrak{b}}\left(\begin{smallmatrix}1&a\\0&1\end{smallmatrix}\right),\sigma_{\mathfrak{b}}^{-1})}\\ &= 1,\end{aligned}$$

since by (3.2), (3.6), and (3.3), we have

$$\sigma_\kappa(\sigma_{\mathfrak{b}}\left(\begin{smallmatrix}1&a\\0&1\end{smallmatrix}\right),\sigma_{\mathfrak{b}}^{-1}) = \frac{\sigma_\kappa(\sigma_{\mathfrak{b}}\left(\begin{smallmatrix}1&a\\0&1\end{smallmatrix}\right)\sigma_{\mathfrak{b}}^{-1},I)\sigma_\kappa(\sigma_{\mathfrak{b}},\sigma_{\mathfrak{b}}^{-1})}{\sigma_\kappa(\sigma_{\mathfrak{b}}\left(\begin{smallmatrix}1&a\\0&1\end{smallmatrix}\right)\sigma_{\mathfrak{b}}^{-1},\sigma_{\mathfrak{b}})} = 1.$$

Furthermore, we have by (3.2) twice and (3.6), that

$$\begin{aligned}\overline{\sigma_\kappa(\sigma_\mathfrak{a}\omega_{c,d}\sigma_\mathfrak{b}^{-1},\sigma_\mathfrak{b}\left(\begin{smallmatrix}1&n\\0&1\end{smallmatrix}\right)\sigma_\mathfrak{b}^{-1})} &= \frac{\sigma_\kappa(\sigma_\mathfrak{a},\omega_{c,d}\sigma_\mathfrak{b}^{-1})}{\sigma_\kappa(\sigma_\mathfrak{a},\omega_{c,d}\left(\begin{smallmatrix}1&n\\0&1\end{smallmatrix}\right)\sigma_\mathfrak{b}^{-1})\sigma_\kappa(\omega_{c,d}\sigma_\mathfrak{b}^{-1},\sigma_\mathfrak{b}\left(\begin{smallmatrix}1&n\\0&1\end{smallmatrix}\right)\sigma_\mathfrak{b}^{-1})}\\ &= \frac{\sigma_\kappa(\sigma_\mathfrak{a},\omega_{c,d}\sigma_\mathfrak{b}^{-1})\sigma_\kappa(\omega_{c,d},\sigma_\mathfrak{b}^{-1})}{\sigma_\kappa(\sigma_\mathfrak{a},\omega_{c,d}\left(\begin{smallmatrix}1&n\\0&1\end{smallmatrix}\right)\sigma_\mathfrak{b}^{-1})\sigma_\kappa(\omega_{c,d},\left(\begin{smallmatrix}1&n\\0&1\end{smallmatrix}\right)\sigma_\mathfrak{b}^{-1})\sigma_\kappa(\sigma_\mathfrak{b}^{-1},\sigma_\mathfrak{b}\left(\begin{smallmatrix}1&n\\0&1\end{smallmatrix}\right)\sigma_\mathfrak{b}^{-1})}\\ &= \frac{\sigma_\kappa(\sigma_\mathfrak{a},\omega_{c,d}\sigma_\mathfrak{b}^{-1})\sigma_\kappa(\omega_{c,d},\sigma_\mathfrak{b}^{-1})}{\sigma_\kappa(\sigma_\mathfrak{a},\omega_{c,d}\left(\begin{smallmatrix}1&n\\0&1\end{smallmatrix}\right)\sigma_\mathfrak{b}^{-1})\sigma_\kappa(\omega_{c,d},\left(\begin{smallmatrix}1&n\\0&1\end{smallmatrix}\right)\sigma_\mathfrak{b}^{-1})}.\end{aligned} \tag{3.21}$$

We also have

$$j(\omega_{c,d}\left(\begin{smallmatrix}1&n\\0&1\end{smallmatrix}\right),z) = j(\omega_{c,d},z+n). \tag{3.22}$$

Thus using (3.19), (3.20), (3.21), and (3.22) we find that the contribution from $\omega_{c,d}\left(\begin{smallmatrix}1&n\\0&1\end{smallmatrix}\right)$, $n\in\mathbb{Z}$, is

$$\begin{aligned}\sum_{n\in\mathbb{Z}} &\overline{v(\sigma_\mathfrak{a}\omega_{c,d}\sigma_\mathfrak{b}^{-1})}e(-\eta_\mathfrak{b}n)\sigma_\kappa(\sigma_\mathfrak{a},\omega_{c,d}\sigma_\mathfrak{b}^{-1})\sigma_\kappa(\omega_{c,d},\sigma_\mathfrak{b}^{-1})\\ &\times \operatorname{Im}(\omega_{c,d}(z+n))^s e((m+\eta_\mathfrak{a})\omega_{c,d}(z+n))\left(\frac{j(\omega_{c,d},z+n)}{|j(\omega_{c,d},z+n)|}\right)^{-\kappa}.\end{aligned} \tag{3.23}$$

By using Poisson summation, we find

$$\begin{aligned}&\sum_{n\in\mathbb{Z}} e(-\eta_\mathfrak{b}n)\operatorname{Im}(\omega_{c,d}(z+n))^s e((m+\eta_\mathfrak{a})\omega_{c,d}(z+n))\left(\frac{j(\omega_{c,d},z+n)}{|j(\omega_{c,d},z+n)|}\right)^{-\kappa}\\ =&\sum_{n\in\mathbb{Z}}\int_{-\infty}^{\infty} e(-\eta_\mathfrak{b}t)\operatorname{Im}(\omega_{c,d}(z+t))^s e((m+\eta_\mathfrak{a})\omega_{c,d}(z+t))\left(\frac{j(\omega_{c,d},z+t)}{|j(\omega_{c,d},z+t)|}\right)^{-\kappa} e(-nt)dt.\end{aligned} \tag{3.24}$$

It is in the next step which forces us to always choose $c>0$ in the definition of the Kloosterman sum, despite the fact that the Poincaré series are well-defined modulo ± 1. Let $\omega_{c,d} = \left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$. Then, we have

$$\omega_{c,d}(z+t) = \frac{a}{c} - \frac{1}{c^2(z+t+\frac{d}{c})} \text{ and } j(\omega_{c,d},z+t) = c(z+t+\tfrac{d}{c}).$$

Shifting the integral in (3.24) by $-x-\frac{d}{c}$, we have that (3.24) is equal to

$$\begin{aligned}\sum_{n\in\mathbb{Z}}\int_{-\infty}^{\infty} &e(-\eta_\mathfrak{b}(t-x-\tfrac{d}{c}))\operatorname{Im}\left(\frac{a}{c}-\frac{1}{c^2(t+iy)}\right)^s\\ &\cdot e\left((m+\eta_\mathfrak{a})\left(\frac{a}{c}-\frac{1}{c^2(t+iy)}\right)\right)e^{-i\kappa\arg(t+iy)}e(-n(t-x-\tfrac{d}{c}))dt.\end{aligned} \tag{3.25}$$

On recalling (3.17), we find that (3.24) is further equal to

$$e^{-\frac{\pi i}{2}\kappa}\sum_{n\in\mathbb{Z}} e((n+\eta_\mathfrak{b})x)e\left((m+\eta_\mathfrak{a})\frac{a}{c}+(n+\eta_\mathfrak{b})\frac{d}{c}\right)\frac{y^s}{c^{2s}}B^\kappa(c,m+\eta_\mathfrak{a},n+\eta_\mathfrak{b},y,s). \tag{3.26}$$

By using this in (3.23) and summing over $d \bmod c\mathbb{Z}$ and $c \in \mathcal{C}_{\mathfrak{a},\mathfrak{b}}$, we confirm the proclaimed Fourier expansion. □

Next, we turn our attention to the special case of the Eisenstein series, i.e. $m = \eta_{\mathfrak{a}} = 0$. We need to analyse the integral B^κ further:

$$\begin{aligned} B^\kappa(c,0,n,y,s) &= e^{\frac{\pi i}{2}\kappa} \int_{-\infty}^{\infty} e(-nt) e^{-i\kappa \arg(t+iy)} \frac{dt}{(t^2+y^2)^s} \\ &= y^{1-2s} \int_{-\infty}^{\infty} e(-nyu) e^{-i\kappa \arg(1-iu)} \frac{du}{(1+u^2)^s}. \end{aligned} \tag{3.27}$$

Let us now define

$$B(y) = \int_{-\infty}^{\infty} e(-yu)(1-iu)^{-\frac{\kappa}{2}}(1+iu)^{\frac{\kappa}{2}} \frac{du}{(1+u^2)^s}. \tag{3.28}$$

Then, we have $B^\kappa(c,0,n,y,s) = y^{1-2s}B(ny)$. Letting $t = iu$ we find

$$B(y) = i\int_{i\infty}^{-i\infty} e^{-2\pi yt}(1-t)^{-\frac{\kappa}{2}-s}(1+t)^{\frac{\kappa}{2}-s}dt.$$

Suppose $y > 0$. Then, we make the substitution $u = t-1$ and shifting the contour to a lock-hole integral coming from infinity looping around 0 in positive direction and going back to infinity, we arrive at

$$B(y) = ie^{-2\pi y}\int_{\infty}^{(0^+)} e^{-2\pi yu}(-u)^{-\frac{\kappa}{2}-s}(2+u)^{\frac{\kappa}{2}-s}du.$$

By substituting $t = 2\pi yu$, we find

$$B(y) = ie^{-2\pi y}(2\pi y)^{\frac{\kappa}{2}+s-1}2^{\frac{\kappa}{2}-s}\int_{\infty}^{(0^+)} e^{-t}(-t)^{-\frac{\kappa}{2}-s}\left(1+\frac{t}{4\pi y}\right)^{\frac{\kappa}{2}-s}du.$$

By inserting the definition of the Whittaker function (A.1), we find

$$B(y) = \frac{\pi^s y^{s-1}}{\Gamma(\frac{\kappa}{2}+s)} W_{\frac{\kappa}{2},\frac{1}{2}-s}(4\pi y), \quad \forall y > 0.$$

Similarly, we find

$$B(y) = \frac{\pi^s |y|^{s-1}}{\Gamma(-\frac{\kappa}{2}+s)} W_{-\frac{\kappa}{2},\frac{1}{2}-s}(4\pi |y|), \quad \forall y < 0.$$

By either taking limits or directly from the equation (A.20), we find

$$B(0) = 2^{2-2s}\pi \frac{\Gamma(2s-1)}{\Gamma(s-\frac{\kappa}{2})\Gamma(s+\frac{\kappa}{2})}.$$

By putting this information back into (3.27), we find for $n + \eta_{\mathfrak{b}} \neq 0$

$$B^\kappa(c,0,n+\eta_{\mathfrak{b}},y,s) = \frac{\pi^s|n+\eta_{\mathfrak{b}}|^{s-1}}{\Gamma(s+\operatorname{sign}(n+\eta_{\mathfrak{b}})\frac{\kappa}{2})} y^{-s} W_{\operatorname{sign}(n+\eta_{\mathfrak{b}})\frac{\kappa}{2},\frac{1}{2}-s}(4\pi|n+\eta_{\mathfrak{b}}|y)$$

and

$$B^{\kappa}(c,0,0,y,s) = 2^{2-2s}\pi\frac{\Gamma(2s-1)}{\Gamma(s+\frac{\kappa}{2})\Gamma(s-\frac{\kappa}{2})}y^{1-2s}.$$

We have concluded the following proposition.

Proposition 3.4.6. *The Eisenstein series admit the following Fourier expansion*

$$\begin{aligned}(\mathcal{E}_{\mathfrak{a}}^{\upsilon,\kappa}|_{\kappa}\sigma_{\mathfrak{b}})(z,s) &= \delta_{\mathfrak{a},\mathfrak{b}}y^{s} + \delta_{\eta_{\mathfrak{b}},0}2^{2-2s}\pi\frac{\Gamma(2s-1)}{\Gamma(s-\frac{\kappa}{2})\Gamma(s+\frac{\kappa}{2})}\mathcal{Z}_{\mathfrak{a},\mathfrak{b}}^{\upsilon,\kappa}(0,0;s)y^{1-s}\\ &+ \pi^{s}\sum_{\substack{n\in\mathbb{Z}\\ n+\eta_{\mathfrak{b}}\neq 0}}\frac{|n+\eta_{\mathfrak{b}}|^{s-1}}{\Gamma(s+\operatorname{sign}(n+\eta_{\mathfrak{b}})\frac{\kappa}{2})}\mathcal{Z}_{\mathfrak{a},\mathfrak{b}}^{\upsilon,\kappa}(0,n;s)W_{\operatorname{sign}(n+\eta_{\mathfrak{b}})\frac{\kappa}{2},\frac{1}{2}-s}(4\pi|n+\eta_{\mathfrak{b}}|y)e((n+\eta_{\mathfrak{b}})x),\end{aligned} \tag{3.29}$$

where we recall the Kloosterman zeta function $\mathcal{Z}_{\mathfrak{a},\mathfrak{b}}^{\upsilon,\kappa}$ from (3.14).

Proposition 3.4.7. *The Eisenstein series $\mathcal{E}_{\mathfrak{a}}^{\upsilon,\kappa}(\cdot,s)$ for $\mathrm{Re}(s)>1$ are Maass forms with eigenvalue $s(1-s)$.*

Proof. We have already shown that $\mathcal{E}_{\mathfrak{a}}^{\upsilon,\kappa}(\cdot,s)$ is modular and satisfies the growth conditions at the cusps, thus it suffices to show that is also an analytic eigenfunction with eigenvalue $s(1-s)$. To this end, note that $\mathcal{E}_{\mathfrak{a}}^{\upsilon,\kappa}(\cdot,s)$ is a locally absolutely uniformly convergent linear combination of terms of the shape $y^s|_{\kappa}\gamma$. Now, y^s is a real analytic function, therefore so is $y^s|_{\kappa}\gamma$ and furthermore $\mathcal{E}_{\mathfrak{a}}^{\upsilon}(\cdot,s)$. Finally, we have $-\Delta_{\kappa}y^s = s(1-s)y^s$ and by Lemma 3.4.1 Δ_{κ} commutes with the slash operators. Note that we are allowed to interchange Δ_{κ} with the infinite sum due to the afore mentioned convergence of analytic functions. □

3.5 SPECTRAL THEOREM

The goal of this section is to understand the spectrum of $-\Delta_{\kappa}$. For this matter, we consider its unique self-adjoint extension (which we shall also denote $-\Delta_{\kappa}$) to the Hilbert space $\mathcal{H}_{\kappa}(\Gamma,\upsilon)$, which consists of all equivalence classes of with respect to $y^{-2}dxdy$ measurable functions $f\in\mathcal{F}_{\kappa}(\Gamma,\upsilon)$ that are square-integrable with respect to the inner product

$$\langle f,g\rangle = \int_{\mathcal{F}_{\Gamma}} f(z)\overline{g(z)}\frac{dxdy}{y^2}. \tag{3.30}$$

Where as usual two functions f,g are equivalent if and only if $\|f-g\|_2 = 0$, where $\|f\|_2 = \langle f,f\rangle^{\frac{1}{2}}$ denotes the associated norm. In what follows, we shall not differentiate between a function and its equivalence class. We may also extend the increase and decrease operators K_κ, respectively Λ_κ, to $\mathcal{H}_\kappa(\Gamma, \upsilon)$ in a manner such that the equalities in Lemma 3.4.2 continue to hold.

Lemma 3.5.1. *Let $f, g \in \mathcal{H}_\kappa(\Gamma, \upsilon)$. Then, we have the following equalities*

$$\langle g, -\Delta_\kappa f\rangle = \langle K_\kappa g, K_\kappa f\rangle - \tfrac{\kappa}{2}(1 + \tfrac{\kappa}{2})\langle g, f\rangle,$$
$$\langle g, -\Delta_\kappa f\rangle = \langle \Lambda_\kappa g, \Lambda_\kappa f\rangle + \tfrac{\kappa}{2}(1 - \tfrac{\kappa}{2})\langle g, f\rangle.$$

Proof. See [Roe66, Satz 3.1]. □

It follows from this that all eigenvalues of the Laplace–Beltrami operator $-\Delta_\kappa$ on $\mathcal{H}_\kappa(\Gamma, \upsilon)$ are $\geq \frac{|\kappa|}{2}(1 - \frac{|\kappa|}{2})$. Let us denote with $\mathcal{H}_\kappa(\Gamma, \upsilon, \lambda) = \mathcal{A}_\kappa^\infty(\Gamma, \upsilon, \lambda) \cap \mathcal{H}_\kappa(\Gamma, \upsilon)$ the space of all square-integrable Maass forms with eigenvalue λ.

Lemma 3.5.2. *Let $f \in \mathcal{H}_\kappa(\Gamma, \upsilon)$. Then, we have*

1. $-\Delta_\kappa f = -\frac{\kappa}{2}(1 + \frac{\kappa}{2}) f \Leftrightarrow K_\kappa f = 0 \Leftrightarrow y^{\frac{\kappa}{2}}\overline{f(z)}$ *is holomorphic,*
2. $-\Delta_\kappa f = \frac{\kappa}{2}(1 - \frac{\kappa}{2}) f \Leftrightarrow \Lambda_\kappa f = 0 \Leftrightarrow y^{-\frac{\kappa}{2}} f(z)$ *is holomorphic.*

Proof. This is a combination of [Roe66, Lemma 3.2] and the previous lemma. □

We now consider the subspace of cuspidal functions $\mathcal{C}_\kappa(\Gamma, \upsilon)$, which consists of all functions $f \in \mathcal{H}_\kappa(\Gamma, \upsilon)$ which vanish at every singular cusp, that is for every singular cusp $\mathfrak{a}$ we have

$$\int_0^1 (f|_\kappa \sigma_\mathfrak{a})(z)dx = 0, \quad \text{for almost all } y.$$

The Laplace–Beltrami operator Δ_κ maps the space $\mathcal{C}_\kappa(\Gamma, \upsilon)$ into itself. We are now able to state the spectral theorem for the cuspidal space.

Theorem 3.5.3. *There is an at most countable orthonormal basis of cuspidal Maass forms $\mathcal{B}_\kappa^c(\Gamma, \upsilon) \subseteq \mathcal{A}_\kappa^\infty(\Gamma, \upsilon) \cap \mathcal{C}_\kappa(\Gamma, \upsilon)$ with eigenvalues $\lambda_h \in [\frac{|\kappa|}{2}(1 - \frac{|\kappa|}{2}), \infty[$ for every $h \in \mathcal{B}_\kappa^c(\Gamma, \upsilon)$. Each eigenvalue appears with finite multiplicity and the sum*

$$\sum_{\substack{h \in \mathcal{B}_\kappa^c(\Gamma, \upsilon) \\ \lambda_h \neq 0}} \lambda_h^{-2}$$

converges. For any $f \in \mathcal{C}_\kappa(\Gamma, \upsilon)$ we have the expansion (in norm)

$$f(z) = \sum_{h \in \mathcal{B}_\kappa^c(\Gamma, \upsilon)} \langle f, h\rangle h(z).$$

The right-hand side further converges absolutely uniformly in $z \in \mathbb{H}$.

Proof. See [Roe66, Chapter 8]. □

Let $\mathcal{N}_\kappa(\Gamma, \upsilon) \subseteq \mathcal{H}_\kappa(\Gamma, \upsilon)$ denote the orthogonal complement of $\mathcal{C}_\kappa(\Gamma, \upsilon)$. Before stating the spectral theorem for $\mathcal{N}_\kappa(\Gamma, \upsilon)$, we need to collect a couple more facts.

Proposition 3.5.4. *The Eisenstein series $\mathcal{E}_\mathfrak{a}^{\upsilon,\kappa}(z_0, s)$ and its Fourier coefficients (3.29) admit meromorphic continuation to the whole complex plane $s \in \mathbb{C}$ with at most simple poles in $[0, \frac{1}{2}[\cup]\frac{1}{2}, 1]$. If $s_0 \in]\frac{1}{2}, 1]$ is such a pole, then s_0 is also a simple pole of $\mathcal{E}_\mathfrak{a}^{\upsilon,\kappa}(z, s)$ for every $z \in \mathbb{H}$, and a simple pole of its 0-th Fourier at the cusp $\mathfrak{a}$, in other words a simple pole of*

$$\frac{\Gamma(2s-1)}{\Gamma(s-\frac{\kappa}{2})\Gamma(s+\frac{\kappa}{2})}\mathcal{Z}_{\mathfrak{a},\mathfrak{a}}^{\upsilon,\kappa}(0, 0; s).$$

For s not a pole, the Eisenstein series $\mathcal{E}_\mathfrak{a}^{\upsilon,\kappa}(\cdot, s)$ are Maass forms and their Fourier expansions (3.29) continue to hold.

Proof. See [Roe66, Chapters 10 & 11]. □

Let $s_0 \in]\frac{1}{2}, 1]$ be a pole of $\mathcal{E}_\mathfrak{a}^{\upsilon}(z, s)$. Then, the residual function

$$f(z) = \operatorname*{Res}_{s=s_0} \mathcal{E}_\mathfrak{a}^{\upsilon,\kappa}(z, s) \tag{3.31}$$

is a square-integrable Maass form with eigenvalue $s_0(1 - s_0)$, that does not vanish at the cusp $\mathfrak{a}$. The span of these non-cuspidal Maass forms is called the residual spectrum. We are now able to state the spectral theorem for the space $\mathcal{N}_\kappa(\Gamma, \upsilon)$.

Theorem 3.5.5. *There is a complete finite set of orthonormal eigenfunctions $\mathcal{B}_\kappa^r(\Gamma, \upsilon)$ and eigenpackets $\mathcal{P}_\kappa(\Gamma, \upsilon)$. The eigenfunctions are given by a collection of normalised residual Maass forms (3.31) and the eigenpackets are given by the Eisenstein series attached to singular cusps. For any $f \in \mathcal{N}_\kappa(\Gamma, \upsilon)$ we have the expansion (in norm)*

$$f(z) = \sum_{h \in \mathcal{B}_\kappa^r(\Gamma,\upsilon)} \langle f, h \rangle h(z) + \frac{1}{4\pi} \sum_{\mathfrak{a} \text{ sing.}} \int_{-\infty}^{\infty} \langle f, \mathcal{E}_\mathfrak{a}^{\upsilon,\kappa}(\cdot, \tfrac{1}{2} + ir) \rangle \mathcal{E}_\mathfrak{a}^{\upsilon,\kappa}(z, \tfrac{1}{2} + ir) dr.$$

Here, $\langle f, \mathcal{E}_\mathfrak{a}^{\upsilon,\kappa}(\cdot, \frac{1}{2} + ir) \rangle$ is to be understood as in (3.30) if the integral converges absolutely, otherwise as the limit

$$\lim_{g \to f} \langle g, \mathcal{E}_\mathfrak{a}^{\upsilon,\kappa}(\cdot, \tfrac{1}{2} + ir) \rangle,$$

where the limit is taken over the functions $g \in \mathcal{H}_\kappa(\Gamma, \upsilon)$ with compact support in $\mathcal{F}_\Gamma$ which converge in norm to f. The integral over r with respect to z in a compact subset $\mathcal{K} \subseteq \mathbb{H}$ is absolutely convergent. In other words,

$$\lim_{\mu \to \infty} \int_{-\mu}^{\mu} |\ldots| dr$$

converges uniformly with respect to $z \in \mathcal{K}$.

Proof. See [Roe66, Chapter 12]. □

Let us set $\mathcal{B}_\kappa(\Gamma, \upsilon) = \mathcal{B}^c_\kappa(\Gamma, \upsilon) \cup \mathcal{B}^r_\kappa(\Gamma, \upsilon)$. Then, $\mathcal{B}_\kappa(\Gamma, \upsilon)$ together with $\mathcal{P}_\kappa(\Gamma, \upsilon)$ form a complete set of eigenfunctions and eigenpackets for the space $\mathcal{H}_\kappa(\Gamma, \upsilon)$. As a consequence, we have the following Parseval identity.

Proposition 3.5.6. *For $f, g \in \mathcal{H}_\kappa(\Gamma, \upsilon)$ we have*

$$\langle f, g\rangle = \sum_{h \in \mathcal{B}_\kappa(\Gamma,\upsilon)} \langle f, h\rangle \overline{\langle g, h\rangle} + \frac{1}{4\pi} \sum_{\mathfrak{a} \text{ sing.}} \int_{-\infty}^{\infty} \langle f, \mathcal{E}^{\upsilon,\kappa}_{\mathfrak{a}}(\cdot, \tfrac{1}{2} + ir)\rangle \overline{\langle g, \mathcal{E}^{\upsilon,\kappa}_{\mathfrak{a}}(\cdot, \tfrac{1}{2} + ir)\rangle} dr. \quad (3.32)$$

Proof. This follows from the previous spectral theorems in combination with [Roe66, Lemma 5.2 & Eqs. (12.15)-(12.21)] and the polarisation identity. □

3.6 PRE-TRACE FORMULAE

The goal of this section is to derive so-called pre-trace formulae, which shall then be used in Section 3.10 to derive the Kuznetsov trace formula. We follow the method originally developed by Kuznetsov [Kuz80] and evaluate inner products with Poincaré series in two ways. The results we shall state here are generalisations of the work of Deshouillers–Iwaniec [DI83], Ahlgren–Andersen [AA18], and Proskurin [Pro05, Pro79]. We will make frequent use of estimates of the latter two references. We shall also point out a second method to develop these pre-trace formulae, which is based on the meromorphic continuation of the Kloosterman zeta function $\mathcal{Z}^{\upsilon}_{\mathfrak{a},\mathfrak{b}}(m, n; s)$. This method is employed for example in [Iwa02].

Proposition 3.6.1. *The Poincaré series $\mathcal{U}^{\upsilon,\kappa}_{\mathfrak{a},m}(\cdot, s)$ with $m + \eta_{\mathfrak{a}} > 0$ and $\operatorname{Re}(s) > 1$ are elements of $\mathcal{H}_\kappa(\Gamma, \upsilon)$.*

Proof. By taking Proposition 3.4.4 into account, it is sufficient to prove that the Poincaré series are square-integrable.

Recall the Fourier expansion of the Poincaré series (3.16) and the integral B^κ from (3.17). By shifting the contour integral in B^κ from the real line to $\operatorname{Im} t = -\frac{1}{2}\operatorname{sign}(n + \eta_{\mathfrak{b}})y$ as in [Pro05], one finds

$$B^\kappa(c, m + \eta_{\mathfrak{a}}, n + \eta_{\mathfrak{b}}, y, s) \ll_{A,B} e^{-\pi|n+\eta_{\mathfrak{b}}|y} y^{1-2\operatorname{Re}(s)}, \quad \forall \operatorname{Re}(s) > \frac{1}{2} + A, |\operatorname{Im}(s)| \leq B. \quad (3.33)$$

The estimate (3.13) further shows

$$\sum_{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}} \frac{|S^{\upsilon,\kappa}_{\mathfrak{a},\mathfrak{b}}(m,n;c)|}{c^{2\operatorname{Re}(s)}} \ll_{\mathfrak{a},\mathfrak{b},A} 1, \quad \forall \operatorname{Re}(s) > 1 + A.$$

Hence, we conclude from Proposition 3.4.5 that

$$\mathcal{U}^{\upsilon,\kappa}_{\mathfrak{a},m}(\sigma_{\mathfrak{b}} z, s) \ll_{A,B,\mathfrak{a},\mathfrak{b}} y^{1-\operatorname{Re}(s)}, \quad \forall \operatorname{Re}(s) > 1 + A, |\operatorname{Im}(s)| < B. \tag{3.34}$$

It easily follows from this that $\mathcal{U}^{\upsilon,\kappa}_{\mathfrak{a},m}(\cdot, s)$ is square-integrable. □

Proposition 3.6.2. *Let $h \in \mathcal{B}_\kappa(\Gamma, \upsilon)$, $\mathfrak{c}$ a singular cusp with respect to Γ and υ, $r \in \mathbb{R}$ and $m + \eta_{\mathfrak{a}} > 0$. Then, for $\operatorname{Re}(s) > 1$ we have*

$$\langle \mathcal{U}^{\upsilon,\kappa}_{\mathfrak{a},m}(\cdot, s), h\rangle = \overline{\rho_h(\mathfrak{a}, m)}(4\pi(m+\eta_{\mathfrak{a}}))^{1-s} \frac{\Gamma(s-\frac{1}{2}-it_h)\Gamma(s-\frac{1}{2}+it_h)}{\Gamma(s-\frac{\kappa}{2})}$$

and

$$\begin{aligned}&\langle \mathcal{U}^{\upsilon,\kappa}_{\mathfrak{a},m}(\cdot, s), \mathcal{E}^{\upsilon,\kappa}_{\mathfrak{c}}(\cdot, \tfrac{1}{2}+ir)\rangle \\ &= \pi^{\frac{1}{2}-ir}(4\pi(m+\eta_{\mathfrak{a}}))^{1-s}(m+\eta_{\mathfrak{a}})^{-\frac{1}{2}-ir}\overline{\mathcal{Z}^{\upsilon,\kappa}_{\mathfrak{c},\mathfrak{a}}(0, m; \tfrac{1}{2}+ir)}\frac{\Gamma(s-\frac{1}{2}+ir)\Gamma(s-\frac{1}{2}-ir)}{\Gamma(s-\frac{\kappa}{2})\Gamma(\frac{1}{2}+\frac{\kappa}{2}-ir)}.\end{aligned}$$

Proof. We have that $\langle \mathcal{U}^{\upsilon,\kappa}_{\mathfrak{a},m}(\cdot, s), h\rangle$ equals

$$\int_{\mathcal{F}_\Gamma} \sum_{\gamma\in\hat{\Gamma}_{\mathfrak{a}}\backslash\hat{\Gamma}} \overline{\upsilon(\gamma)\sigma_\kappa(\sigma_{\mathfrak{a}}^{-1},\gamma)} \operatorname{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^s e((m+\eta_{\mathfrak{a}})\sigma_{\mathfrak{a}}^{-1}\gamma z) \left(\frac{j(\sigma_{\mathfrak{a}}^{-1}\gamma, z)}{|j(\sigma_{\mathfrak{a}}^{-1}\gamma, z)|}\right)^{-\kappa} \overline{h(z)}\frac{dxdy}{y^2}.$$

We recall that the Maass form h has at most polynomial growth at each cusp. In fact, it must satisfy $o(\operatorname{Im}(\sigma_{\mathfrak{b}}^{-1}z)^{\frac{1}{2}})$ at every cusp $\mathfrak{b}$ in order to be square-integrable. On the other hand the Poincaré series satisfy the bound $o(1)$ at every cusp (3.34). These facts together with the absolute locally uniform convergence of the sum (see Proposition 3.4.4) allow us to interchange the integral with the infinite sum. By inserting a further change of variables, we arrive at

$$\begin{aligned}\sum_{\gamma\in\hat{\Gamma}_{\mathfrak{a}}\backslash\hat{\Gamma}} \int_{\sigma_{\mathfrak{a}}^{-1}\gamma\mathcal{F}_{\hat{\Gamma}}} \overline{\upsilon(\gamma)\sigma_\kappa(\sigma_{\mathfrak{a}}^{-1},\gamma)} e((m+\eta_{\mathfrak{a}})z) \left(\frac{j(\sigma_{\mathfrak{a}}^{-1}\gamma, \gamma^{-1}\sigma_{\mathfrak{a}}z)}{|j(\sigma_{\mathfrak{a}}^{-1}\gamma, \gamma^{-1}\sigma_{\mathfrak{a}}z)|}\right)^{-\kappa} \overline{h(\gamma^{-1}\sigma_{\mathfrak{a}}z)}\frac{dxdy}{y^{2-s}} \\ = \int_{\bigcup_{\gamma\in\hat{\Gamma}_{\mathfrak{a}}\backslash\hat{\Gamma}}\sigma_{\mathfrak{a}}^{-1}\gamma\mathcal{F}_{\hat{\Gamma}}} e((m+\eta_{\mathfrak{a}})z)\overline{(h|_\kappa\sigma_{\mathfrak{a}})(z)}\frac{dxdy}{y^{2-s}}.\end{aligned} \tag{3.35}$$

In the second line, we have used the modularity of h and the following equality

$$\begin{aligned}
&\upsilon(\gamma)\sigma_\kappa(\sigma_{\mathfrak{a}}^{-1},\gamma)\left(\frac{j(\sigma_{\mathfrak{a}}^{-1}\gamma,\gamma^{-1}\sigma_{\mathfrak{a}}z)}{|j(\sigma_{\mathfrak{a}}^{-1}\gamma,\gamma^{-1}\sigma_{\mathfrak{a}}z)|}\right)^{\kappa}\upsilon(\gamma^{-1})\left(\frac{j(\gamma^{-1},\sigma_{\mathfrak{a}}z)}{|j(\gamma^{-1},\sigma_{\mathfrak{a}}z)|}\right)^{\kappa}\left(\frac{j(\sigma_{\mathfrak{a}},z)}{|j(\sigma_{\mathfrak{a}},z)|}\right)^{\kappa}\\
=&\overline{\sigma_\kappa(\gamma,\gamma^{-1})}\sigma_\kappa(\sigma_{\mathfrak{a}}^{-1},\gamma)\left(\frac{j(\sigma_{\mathfrak{a}}^{-1}\gamma,\gamma^{-1}\sigma_{\mathfrak{a}}z)}{|j(\sigma_{\mathfrak{a}}^{-1}\gamma,\gamma^{-1}\sigma_{\mathfrak{a}}z)|}\right)^{\kappa}\left(\frac{j(\gamma^{-1}\sigma_{\mathfrak{a}},z)}{|j(\gamma^{-1}\sigma_{\mathfrak{a}},z)|}\right)^{\kappa}\sigma_\kappa(\gamma^{-1},\sigma_{\mathfrak{a}})\\
=&\overline{\sigma_\kappa(\gamma,\gamma^{-1})}\sigma_\kappa(\sigma_{\mathfrak{a}}^{-1},\gamma)\sigma_\kappa(\sigma_{\mathfrak{a}}^{-1}\gamma,\gamma^{-1}\sigma_{\mathfrak{a}})\sigma_\kappa(\gamma^{-1},\sigma_{\mathfrak{a}})\\
=&\overline{\sigma_\kappa(\gamma,\gamma^{-1})}\sigma_\kappa(\sigma_{\mathfrak{a}}^{-1},\sigma_{\mathfrak{a}})\sigma_\kappa(\gamma,\gamma^{-1}\sigma_{\mathfrak{a}})\sigma_\kappa(\gamma^{-1},\sigma_{\mathfrak{a}})\\
=&\overline{\sigma_\kappa(\gamma,\gamma^{-1})}\sigma_\kappa(I,\sigma_{\mathfrak{a}})\sigma_\kappa(\gamma,\gamma^{-1})\\
=&1.
\end{aligned}$$

Now, $\bigcup_{\gamma\in\hat{\Gamma}_{\mathfrak{a}}\backslash\hat{\Gamma}}\sigma_{\mathfrak{a}}^{-1}\gamma\mathcal{F}_{\hat{\Gamma}}$ is a fundamental domain for $\hat{\Gamma}_\infty$ and we may assume it is $\{z\in\mathbb{H}|0\le\mathrm{Re}(z)\le 1\}$. Hence, (3.35) is equal to

$$\int_0^\infty\int_0^1 e((m+\eta_{\mathfrak{a}})z)\overline{(h|_\kappa\sigma_{\mathfrak{a}})(z)}y^{s-2}dxdy.$$

By using the Fourier expansion of h at the cusp $\mathfrak{a}$ (3.15) and exchanging summation with the integral over x, which we may due to the convergence of the Fourier expansion, we find that only the $(m+\eta_{\mathfrak{a}})$-th Fourier coefficient survives the integral over x and thus (3.35) is further equal to

$$\overline{\rho_h(\mathfrak{a},m)}\int_0^\infty e^{-2\pi(m+\eta_{\mathfrak{a}})y}W_{\frac{\kappa}{2},it_h}(4\pi(m+\eta_{\mathfrak{a}})y)y^{s-2}dy.$$

For $\mathrm{Re}(s)>\frac{|\kappa|}{2}$, we have $\mathrm{Re}(s)>\mathrm{Re}(it_h)-\frac{1}{2}$ and thus we may use (A.5) to evaluate the latter integral:

$$\overline{\rho_h(\mathfrak{a},m)}(4\pi(m+\eta_{\mathfrak{a}}))^{1-s}\frac{\Gamma(s-\frac{1}{2}-it_h)\Gamma(s-\frac{1}{2}+it_h)}{\Gamma(s-\frac{\kappa}{2})}.$$

Hence, we have proved the equality for $\mathrm{Re}(s)>\max\{\frac{|\kappa|}{2},1\}$. Now, the right-hand side admits a meromorphic continuation to $\mathrm{Re}(s)>1$ and the left-hand side admits an analytic continuation to $\mathrm{Re}(s)>1$. Hence, the singularities must be removable and we have proved the identity for $\mathrm{Re}(s)>1$. In order to evaluate $\langle\mathcal{U}_{\mathfrak{a},m}^{\upsilon,\kappa}(\cdot,s),\mathcal{E}_{\mathfrak{c}}^{\upsilon,\kappa}(\cdot,\frac{1}{2}+ir)\rangle$, we may proceed as before, since the Eisenstein series satisfy $(\mathcal{E}_{\mathfrak{c}}^{\upsilon,\kappa}|\sigma_{\mathfrak{b}})(z,\frac{1}{2}+ir)=O_r(y^{\frac{1}{2}}\log(y))$ as $y\to\infty$ for every cusp $\mathfrak{b}$, which follows from the Fourier expansion (3.29). We arrive at

$$\begin{aligned}
&\pi^{\frac{1}{2}-ir}\frac{(m+\eta_{\mathfrak{a}})^{-\frac{1}{2}-ir}}{\Gamma(\frac{1}{2}+\frac{\kappa}{2}-ir)}\overline{\mathcal{Z}_{\mathfrak{c},\mathfrak{a}}^{\upsilon,\kappa}(0,m;\tfrac{1}{2}+ir)}\int_0^\infty e^{-2\pi(m+\eta_{\mathfrak{a}})y}W_{\frac{\kappa}{2},-ir}(4\pi(m+\eta_{\mathfrak{a}})y)y^{s-2}dy\\
&=\pi^{\frac{1}{2}-ir}(4\pi(m+\eta_{\mathfrak{a}}))^{1-s}(m+\eta_{\mathfrak{a}})^{-\frac{1}{2}-ir}\overline{\mathcal{Z}_{\mathfrak{c},\mathfrak{a}}^{\upsilon,\kappa}(0,m;\tfrac{1}{2}+ir)}\frac{\Gamma(s-\frac{1}{2}+ir)\Gamma(s-\frac{1}{2}-ir)}{\Gamma(s-\frac{\kappa}{2})\Gamma(\frac{1}{2}+\frac{\kappa}{2}-ir)}.
\end{aligned}$$

□

Remark 3.6.3. Another way to see that the singularities on the right-hand side are removable is to classify all the eigenvalues in $[\frac{|\kappa|}{2}(1-\frac{|\kappa|}{2}),0[$. This has been done in [Roe66, Satz 5.4].

Naturally, one may ask how a similar proposition for negative Fourier coefficients may look like. To this end, we shall employ some trickery. A simple computation shows that for $m+\eta_{\mathfrak{a}}<0$ we have that $\overline{\mathcal{U}^{\overline{\upsilon},-\kappa}_{\mathfrak{a},-m-\delta^{ns}_{\mathfrak{a}}}}(\cdot,s)$ is well-defined and lives in the space $\mathcal{H}_\kappa(\Gamma,\upsilon)$. Therefore, a simple complex conjugation of Proposition 3.6.2 shows the following.

Proposition 3.6.4. *Let $h\in\mathcal{B}_\kappa(\Gamma,\upsilon)$, $\mathfrak{c}$ a singular cusp with respect to Γ and υ, $r\in\mathbb{R}$ and $m+\eta_{\mathfrak{a}}<0$. Then, for $\mathrm{Re}(s)>1$ we have*

$$\langle\overline{\mathcal{U}^{\overline{\upsilon},-\kappa}_{\mathfrak{a},-m-\delta^{ns}_{\mathfrak{a}}}}(\cdot,s),h\rangle=\overline{\rho_h(\mathfrak{a},m)}(4\pi|m+\eta_{\mathfrak{a}}|)^{1-\overline{s}}\frac{\Gamma(\overline{s}-\frac{1}{2}-it_h)\Gamma(\overline{s}-\frac{1}{2}+it_h)}{\Gamma(\overline{s}+\frac{\kappa}{2})}$$

and

$$\begin{aligned}&\langle\overline{\mathcal{U}^{\overline{\upsilon},-\kappa}_{\mathfrak{a},-m-\delta^{ns}_{\mathfrak{a}}}}(\cdot,s),\mathcal{E}^{\upsilon,\kappa}_{\mathfrak{c}}(\cdot,\tfrac{1}{2}+ir)\rangle\\&\quad=\pi^{\frac{1}{2}-ir}(4\pi|m+\eta_{\mathfrak{a}}|)^{1-\overline{s}}|m+\eta_{\mathfrak{a}}|^{-\frac{1}{2}-ir}\overline{\mathcal{Z}^{\upsilon,\kappa}_{\mathfrak{c},\mathfrak{a}}(0,m;\tfrac{1}{2}+ir)}\frac{\Gamma(\overline{s}-\frac{1}{2}+ir)\Gamma(\overline{s}-\frac{1}{2}-ir)}{\Gamma(\overline{s}+\frac{\kappa}{2})\Gamma(\frac{1}{2}-\frac{\kappa}{2}-ir)}.\end{aligned}$$

Proof. All that is left to note is $\eta^{\upsilon}_{\mathfrak{a}}+\eta^{\overline{\upsilon}}_{\mathfrak{a}}=\delta^{ns}_{\mathfrak{a}}$ and $(\overline{h}|_{-\kappa}\sigma_{\mathfrak{a}})=\overline{(h|_{\kappa}\sigma_{\mathfrak{a}})}$. Therefore, $\rho_{\overline{h}}(\mathfrak{a},-m-\delta^{ns}_{\mathfrak{a}})=\overline{\rho_h(\mathfrak{a},m)}$. For the Eisenstein series, we find similarly $\overline{\mathcal{E}^{\upsilon,\kappa}_{\mathfrak{c}}(z,\frac{1}{2}+ir)}=\mathcal{E}^{\overline{\upsilon},-\kappa}_{\mathfrak{c}}(z,\frac{1}{2}-ir)$ and $\mathcal{Z}^{\overline{\upsilon},-\kappa}_{\mathfrak{c},\mathfrak{a}}(0,-m-\delta^{ns}_{\mathfrak{a}};\frac{1}{2}-ir)=\overline{\mathcal{Z}^{\upsilon,\kappa}_{\mathfrak{c},\mathfrak{a}}(0,m;\frac{1}{2}+ir)}$. □

Corollary 3.6.5. *The series $\mathcal{U}^{\upsilon,\kappa}_{\mathfrak{a},m}(z_0,s)$ for $m+\eta_{\mathfrak{a}}>0$ has a meromorphic continuation to all of $s\in\mathbb{C}$. If s_0 is a pole of $\mathcal{U}^{\upsilon,\kappa}_{\mathfrak{a},m}(z_0,s)$, then it is also a pole of $\mathcal{U}^{\upsilon,\kappa}_{\mathfrak{a},m}(z,s)$ for every $z\in\mathbb{H}$. Furthermore, the Fourier coefficients (3.16) of $\mathcal{U}^{\upsilon,\kappa}_{\mathfrak{a},m}(z,s)$ admit a meromorphic continuation to all of $s\in\mathbb{C}$ and the equality (3.16) continues to hold for s not a pole. An analogous statement holds for $\overline{\mathcal{U}^{\overline{\upsilon},-\kappa}_{\mathfrak{a},-m-\delta^{ns}_{\mathfrak{a}}}}$ when $m+\eta_{\mathfrak{a}}<0$.*

Proof. All of this follows from Proposition 3.6.2, respectively Proposition 3.6.4, and the spectral expansion stemming from the conjunction of the Theorems 3.5.3 and 3.5.5. □

Proposition 3.6.6. *Let $m+\eta_{\mathfrak{a}},n+\eta_{\mathfrak{b}}>0$ and $\mathrm{Re}(s_1),\mathrm{Re}(s_2)>1$. Then, we have*

$$\begin{aligned}\langle\mathcal{U}^{\upsilon,\kappa}_{\mathfrak{a},m}(\cdot,s_1),\mathcal{U}^{\upsilon,\kappa}_{\mathfrak{b},n}(\cdot,\overline{s_2})\rangle&=\delta_{\mathfrak{a},\mathfrak{b}}\delta_{m,n}\frac{\Gamma(s_1+s_2-1)}{(2\pi(m+\eta_{\mathfrak{a}}+n+\eta_{\mathfrak{b}}))^{s_1+s_2-1}}-i\left(\frac{m+\eta_{\mathfrak{a}}}{n+\eta_{\mathfrak{b}}}\right)^{\frac{s_2-s_1}{2}}\\&\times 2^{3-s_1-s_2}\sum_{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}}\frac{S^{\upsilon,\kappa}_{\mathfrak{a},\mathfrak{b}}(m,n;c)}{c^{s_1+s_2}}\int_L K_{s_1-s_2}\left(\frac{4\pi\sqrt{(m+\eta_{\mathfrak{a}})(n+\eta_{\mathfrak{b}})}}{c}q\right)\left(q+\frac{1}{q}\right)^{s_1+s_2-2}q^{\kappa-1}dq,\end{aligned}$$

where L is the contour along the semicircle $|q|=1$ from $-i$ to i with $\mathrm{Re}(q)>0$.

Proof. This proposition is essentially the same as [Pro05, Lemma 1] and we follow its proof closely. By unfolding the second Poincaré series (as in the beginning of Proposition 3.6.2), we find that $\langle \mathcal{U}^{\upsilon,\kappa}_{\mathfrak{a},m}(\cdot, s_1), \mathcal{U}^{\upsilon,\kappa}_{\mathfrak{b},n}(\cdot, \overline{s_2})\rangle$ is equal to

$$\int_0^\infty \int_0^1 (\mathcal{U}^{\upsilon,\kappa}_{\mathfrak{a},m}|_\kappa \sigma_\mathfrak{a})(z, s_1)\overline{e((n+\eta_\mathfrak{b})z)}y^{s_2-2}dxdy.$$

By inserting the Fourier expansion of $(\mathcal{U}^{\upsilon,\kappa}_{\mathfrak{a},m}|_\kappa \sigma_\mathfrak{a})(z, s_1)$ (see Proposition 3.4.5) and interchanging the integral over x with the summation, we further evaluate this to

$$\delta_{\mathfrak{a},\mathfrak{b}}\delta_{m,n}\frac{\Gamma(s_1+s_2-1)}{(2\pi(m+\eta_\mathfrak{a}+n+\eta_\mathfrak{b}))^{s_1+s_2-1}} + \int_0^\infty \left(\sum_{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}} \frac{S^{\upsilon,\kappa}_{\mathfrak{a},\mathfrak{b}}(m,n;c)}{c^{2s_1}} B^\kappa(c, m+\eta_\mathfrak{a}, n+\eta_\mathfrak{b}, y, s_1)\right) y^{s_1+s_2-2}e^{-2\pi(n+\eta_\mathfrak{b})y}dy.$$

Due to (3.33), we may further exchange summation and integral assuming $\mathrm{Re}(s_2) > \mathrm{Re}(s_1)$. The integral

$$\int_0^\infty B^\kappa(c, m+\eta_\mathfrak{a}, n+\eta_\mathfrak{b}, y, s_1)y^{s_1+s_2-2}e^{-2\pi(n+\eta_\mathfrak{b})y}dy$$

has been evaluated as[c]

$$-i2^{3-s_1-s_2}\frac{1}{c^{s_2-s_1}}\left(\frac{m+\eta_\mathfrak{a}}{n+\eta_\mathfrak{b}}\right)^{\frac{s_2-s_1}{2}} \times \int_L K_{s_1-s_2}\left(\frac{4\pi\sqrt{(m+\eta_\mathfrak{a})(n+\eta_\mathfrak{b})}}{c}q\right)\left(q+\frac{1}{q}\right)^{s_1+s_2-2} q^{\kappa-1}dq$$

in [Pro05, Eqs. (25),(26)]. This concludes the proposition for $\mathrm{Re}(s_2) > \mathrm{Re}(s_1)$. The case $\mathrm{Re}(s_1) > \mathrm{Re}(s_2)$ follows by symmetry in combination with Proposition 3.3.2 and, finally, the case $\mathrm{Re}(s_1) = \mathrm{Re}(s_2)$ by continuity. □

Proposition 3.6.7. *Let* $m+\eta_\mathfrak{a} > 0$, $n+\eta_\mathfrak{b} < 0$, *and* $\mathrm{Re}(s_1), \mathrm{Re}(s_2) > 1$. *Then, we have*

$$\langle \mathcal{U}^{\upsilon,\kappa}_{\mathfrak{a},m}(\cdot, s_1), \overline{\mathcal{U}^{\overline{\upsilon},-\kappa}_{\mathfrak{b},-n-\delta^{ns}_\mathfrak{b}}(\cdot, s_2)}\rangle = \pi 2^{3-s_1-s_2}\frac{\Gamma(s_1+s_2-1)}{\Gamma(s_1-\frac{\kappa}{2})\Gamma(s_2+\frac{\kappa}{2})}\left(\frac{m+\eta_\mathfrak{a}}{-n-\eta_\mathfrak{b}}\right)^{\frac{s_2-s_1}{2}} \times \sum_{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}} \frac{S^{\upsilon,\kappa}_{\mathfrak{a},\mathfrak{b}}(m,n;c)}{c^{s_1+s_2}} K_{s_1-s_2}\left(4\pi\frac{\sqrt{-(n+\eta_\mathfrak{b})(m+\eta_\mathfrak{a})}}{c}\right)$$

Proof. As in Proposition 3.6.6, we unfold the first Poincaré series and find

$$\langle \mathcal{U}^{\upsilon,\kappa}_{\mathfrak{a},m}(\cdot, s_1), \overline{\mathcal{U}^{\overline{\upsilon},-\kappa}_{\mathfrak{b},-n-\delta^{ns}_\mathfrak{b}}(\cdot, s_2)}\rangle = \int_0^\infty \int_0^1 y^{s_1-2}e((m+\eta_\mathfrak{a})z)(\mathcal{U}^{\overline{\upsilon},-\kappa}_{\mathfrak{b},-n-\delta^{ns}_\mathfrak{b}}|_{-\kappa}\sigma_\mathfrak{a})(z, s_2)dxdy. \tag{3.36}$$

c Recall that our definition of B^κ contains an additional factor of $e^{\frac{\pi i}{2}\kappa}$.

By inserting the Fourier expansion for the Poincaré series, see Proposition 3.4.5, together with Proposition 3.3.2, and exchanging summation with integral, we find that (3.36) further equals

$$\int_0^\infty y^{s_1+s_2-2}e^{-2\pi(m+\eta_{\mathfrak{a}}^v)y}\sum_{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}}\frac{S_{\mathfrak{a},\mathfrak{b}}^{v,\kappa}(m,n;c)}{c^{2s_2}}B^{-\kappa}(c,-n-\eta_{\mathfrak{b}}^v,-m-\eta_{\mathfrak{a}}^v,y,s_2)dy. \quad (3.37)$$

For $\mathrm{Re}(s_1) > \mathrm{Re}(s_2)$, we are allowed to exchange summation and integral once more, thus it remains to evaluate the integral

$$\int_0^\infty y^{s_1+s_2-2}e^{-2\pi(m+\eta_{\mathfrak{a}})y}B^{-\kappa}(c,-n-\eta_{\mathfrak{b}}^v,-m-\eta_{\mathfrak{a}}^v,y,s_2)dy. \quad (3.38)$$

By inserting the definition (3.17) of the integral $B^{-\kappa}$, substituting $t = uy$, and interchanging the integrals, which is allowed in view of their absolute convergence, we arrive at

$$e^{-\frac{\pi i}{2}\kappa}\int_{-\infty}^\infty e^{i\kappa\arg(u+i)}(u^2+1)^{-s_2}\int_0^\infty y^{s_1-s_2-1}\exp\left(-2\pi\left(\frac{-n-\eta_{\mathfrak{b}}^v}{c^2y(1-iu)}+(m+\eta_{\mathfrak{a}}^v)y(1-iu)\right)\right)dydu. \quad (3.39)$$

By making use of the integral representation (A.15), we find that the inner integral is equal to

$$2\left(\sqrt{\frac{-n-\eta_{\mathfrak{b}}^v}{m+\eta_{\mathfrak{a}}^v}}\frac{1}{c(1-iu)}\right)^{s_1-s_2}K_{s_2-s_1}\left(4\pi\frac{\sqrt{-(n+\eta_{\mathfrak{b}}^v)(m+\eta_{\mathfrak{b}}^v)}}{c}\right).$$

By inserting this back into (3.39) and using $\arg(u+i) = \frac{\pi}{2} + \arg(1-iu)$, we find that (3.39) is equal to

$$2\left(\frac{-n-\eta_{\mathfrak{b}}^v}{(m+\eta_{\mathfrak{a}}^v)c^2}\right)^{\frac{s_1-s_2}{2}}K_{s_2-s_1}\left(4\pi\frac{\sqrt{-(n+\eta_{\mathfrak{b}}^v)(m+\eta_{\mathfrak{b}}^v)}}{c}\right)\times\int_{-\infty}^\infty(1-iu)^{-s_1+\frac{\kappa}{2}}(1+iu)^{-s_2-\frac{\kappa}{2}}du. \quad (3.40)$$

The latter integral equates to

$$\int_{-\infty}^\infty(1-iu)^{-s_1+\frac{\kappa}{2}}(1+iu)^{-s_2-\frac{\kappa}{2}}du = \pi 2^{2-s_1-s_2}\frac{\Gamma(s_1+s_2-1)}{\Gamma(s_1-\frac{\kappa}{2})\Gamma(s_2+\frac{\kappa}{2})}$$

by (A.20). This proves the proposition in the case $\mathrm{Re}(s_1) > \mathrm{Re}(s_2)$. The case $\mathrm{Re}(s_2) > \mathrm{Re}(s_1)$ is very similar. There, one needs to unfold the second Poincaré series rather than the first. The case $\mathrm{Re}(s_1) = \mathrm{Re}(s_2)$ follows once again from continuity. □

By evaluating $\langle \mathcal{U}^{\upsilon,\kappa}_{\mathfrak{a},m}(\cdot,s_1), \mathcal{U}^{\upsilon,\kappa}_{\mathfrak{b},n}(\cdot,\overline{s_2})\rangle$, respectively $\langle \mathcal{U}^{\upsilon,\kappa}_{\mathfrak{a},m}(\cdot,s_1), \overline{\mathcal{U}^{\overline{\upsilon},-\kappa}_{\mathfrak{b},-n-\delta^{ns}_{\mathfrak{b}}}(\cdot,s_2)}\rangle$, in a second way, by using Parseval's identity (3.32), we arrive at the pre-trace formulae. In order to state them in a compact form, we set

$$\Lambda(s_1,s_2,r) = \Gamma(s_1-\tfrac{1}{2}-ir)\Gamma(s_1-\tfrac{1}{2}+ir)\Gamma(s_2-\tfrac{1}{2}-ir)\Gamma(s_2-\tfrac{1}{2}+ir).$$

Proposition 3.6.8. *Let $\sigma, t\in\mathbb{R}$ with $\sigma>1$ and $m+\eta_{\mathfrak{a}}, n+\eta_{\mathfrak{b}}>0$. Then, we have*

$$\sum_{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}} \frac{S^{\upsilon,\kappa}_{\mathfrak{a},\mathfrak{b}}(m,n;c)}{c^{2\sigma}} \int_L K_{it}\left(4\pi\frac{\sqrt{(m+\eta_{\mathfrak{a}})(n+\eta_{\mathfrak{b}})}}{c}q\right)\left(q+\frac{1}{q}\right)^{2\sigma-2} q^{\kappa-1}dq$$
$$= -i\delta_{\mathfrak{a},\mathfrak{b}}\delta_{m,n}\frac{\Gamma(2\sigma-1)}{4(\pi(m+\eta_{\mathfrak{a}}+n+\eta_{\mathfrak{b}}))^{2\sigma-1}} + i\frac{2^{1-2\sigma}\pi^{2-2\sigma}((m+\eta_{\mathfrak{a}})(n+\eta_{\mathfrak{b}}))^{1-\sigma}}{\Gamma(\sigma-\frac{\kappa}{2}+\frac{it}{2})\Gamma(\sigma-\frac{\kappa}{2}-\frac{it}{2})}$$
$$\times\Bigg\{\sum_{h\in\mathcal{B}_\kappa(\Gamma,\upsilon)}\overline{\rho_h(\mathfrak{a},m)}\rho_h(\mathfrak{b},n)\Lambda(\sigma+\tfrac{it}{2},\sigma-\tfrac{it}{2},t_h) + \frac{1}{4\sqrt{(m+\eta_{\mathfrak{a}})(n+\eta_{\mathfrak{b}})}}$$
$$\times\sum_{\mathfrak{c}\text{ sing.}}\int_{-\infty}^{\infty}\left(\frac{m+\eta_{\mathfrak{a}}}{n+\eta_{\mathfrak{b}}}\right)^{-ir}\frac{\overline{\mathcal{Z}^{\upsilon,\kappa}_{\mathfrak{c},\mathfrak{a}}(0,m;\frac{1}{2}+ir)}}{\Gamma(\frac{1}{2}+\frac{\kappa}{2}-ir)}\frac{\mathcal{Z}^{\upsilon,\kappa}_{\mathfrak{c},\mathfrak{b}}(0,n;\frac{1}{2}+ir)}{\Gamma(\frac{1}{2}+\frac{\kappa}{2}+ir)}\Lambda(\sigma+\tfrac{it}{2},\sigma-\tfrac{it}{2},r)dr\Bigg\}.$$

Proposition 3.6.9. *Let $\sigma, t\in\mathbb{R}$ with $\sigma>1$ and $m+\eta_{\mathfrak{a}}, n+\eta_{\mathfrak{b}}<0$. Then, we have*

$$\sum_{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}} \frac{S^{\upsilon,\kappa}_{\mathfrak{a},\mathfrak{b}}(m,n;c)}{c^{2\sigma}} \int_L K_{it}\left(4\pi\frac{\sqrt{(m+\eta_{\mathfrak{a}})(n+\eta_{\mathfrak{b}})}}{c}q\right)\left(q+\frac{1}{q}\right)^{2\sigma-2} q^{-\kappa-1}dq$$
$$= -i\delta_{\mathfrak{a},\mathfrak{b}}\delta_{m,n}\frac{\Gamma(2\sigma-1)}{4(\pi|m+\eta_{\mathfrak{a}}+n+\eta_{\mathfrak{b}}|)^{2\sigma-1}} + i\frac{2^{1-2\sigma}\pi^{2-2\sigma}((m+\eta_{\mathfrak{a}})(n+\eta_{\mathfrak{b}}))^{1-\sigma}}{\Gamma(\sigma+\frac{\kappa}{2}+\frac{it}{2})\Gamma(\sigma+\frac{\kappa}{2}-\frac{it}{2})}$$
$$\times\Bigg\{\sum_{h\in\mathcal{B}_\kappa(\Gamma,\upsilon)}\overline{\rho_h(\mathfrak{a},m)}\rho_h(\mathfrak{b},n)\Lambda(\sigma+\tfrac{it}{2},\sigma-\tfrac{it}{2},t_h) + \frac{1}{4\sqrt{(m+\eta_{\mathfrak{a}})(n+\eta_{\mathfrak{b}})}}$$
$$\times\sum_{\mathfrak{c}\text{ sing.}}\int_{-\infty}^{\infty}\left(\frac{m+\eta_{\mathfrak{a}}}{n+\eta_{\mathfrak{b}}}\right)^{-ir}\frac{\overline{\mathcal{Z}^{\upsilon,\kappa}_{\mathfrak{c},\mathfrak{a}}(0,m;\frac{1}{2}+ir)}}{\Gamma(\frac{1}{2}-\frac{\kappa}{2}-ir)}\frac{\mathcal{Z}^{\upsilon,\kappa}_{\mathfrak{c},\mathfrak{b}}(0,n;\frac{1}{2}+ir)}{\Gamma(\frac{1}{2}-\frac{\kappa}{2}+ir)}\Lambda(\sigma+\tfrac{it}{2},\sigma-\tfrac{it}{2},r)dr\Bigg\}.$$

Proposition 3.6.10. *Let $\sigma, t\in\mathbb{R}$ with $\sigma>1$ and $m+\eta_{\mathfrak{a}}>0$,$n+\eta_{\mathfrak{b}}<0$. Then, we have*

$$\sum_{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}} \frac{S^{\upsilon,\kappa}_{\mathfrak{a},\mathfrak{b}}(m,n;c)}{c^{2\sigma}} K_{2it}\left(4\pi\frac{\sqrt{(m+\eta_{\mathfrak{a}})|n+\eta_{\mathfrak{b}}|}}{c}\right) = \frac{(2\pi)^{1-2\sigma}((m+\eta_{\mathfrak{a}})|n+\eta_{\mathfrak{b}}|)^{1-\sigma}}{\Gamma(2\sigma-1)}$$
$$\times\Bigg\{\sum_{h\in\mathcal{B}_\kappa(\Gamma,\upsilon)}\overline{\rho_h(\mathfrak{a},m)}\rho_h(\mathfrak{b},n)\Lambda(\sigma+it,\sigma-it,t_h) + \frac{1}{4\sqrt{(m+\eta_{\mathfrak{a}})|n+\eta_{\mathfrak{b}}|}}$$
$$\times\sum_{\mathfrak{c}\text{ sing.}}\int_{-\infty}^{\infty}\left|\frac{m+\eta_{\mathfrak{a}}}{n+\eta_{\mathfrak{b}}}\right|^{-ir}\frac{\overline{\mathcal{Z}^{\upsilon,\kappa}_{\mathfrak{c},\mathfrak{a}}(0,m;\frac{1}{2}+ir)}}{\Gamma(\frac{1}{2}+\frac{\kappa}{2}-ir)}\frac{\mathcal{Z}^{\upsilon,\kappa}_{\mathfrak{c},\mathfrak{b}}(0,n;\frac{1}{2}+ir)}{\Gamma(\frac{1}{2}-\frac{\kappa}{2}+ir)}\Lambda(\sigma+it,\sigma-it,r)dr\Bigg\}.$$

In Section 3.10, we shall derive the Kuznetsov trace formulae from these equations.

3.7 HOLOMORPHIC MODULAR FORMS

On the set of holomorphic functions $f : \mathbb{H} \to \mathbb{C}$, we define the slash operators $|_k\gamma$ for every matrix $\gamma \in \mathrm{SL}_2(\mathbb{R})$ as follows

$$(f|_k\gamma)(z) = j(\gamma, z)^{-k} f(\gamma z).$$

No confusion should arise between the slash operator for Maass forms as their weight is always denoted with κ and the weight of holomorphic forms is denoted by k. The slash operators once more satisfy the equality

$$f|_k\gamma\tau = \sigma_k(\gamma, \tau)(f|_k\gamma)|_k\tau, \quad \forall\gamma, \tau \in \mathrm{SL}_2(\mathbb{R}).$$

Definition 3.7.1. Let Γ be a Fuchsian group of the first kind and υ a multiplier system of weight k with respect to Γ. A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is called *modular* of weight k with respect to υ (and Γ) if it satisfies

$$f|_k\gamma = \upsilon(\gamma) f, \quad \forall\gamma \in \Gamma.$$

Every such function has a Fourier expansion at a cusp $\mathfrak{a}$ of the following type:

$$(f|_k\sigma_{\mathfrak{a}})(z) = \sum_{m \in \mathbb{Z}} \psi_f(\mathfrak{a}, m) e((m + \eta_{\mathfrak{a}})z). \tag{3.41}$$

Definition 3.7.2. Let $f : \mathbb{H} \to \mathbb{C}$ be a holomorphic modular function (of weight k with respect to υ and Γ). If for every cusp $\mathfrak{a}$ of Γ the Fourier expansion at $\mathfrak{a}$ (3.41) may be restricted to those m with $m + \eta_{\mathfrak{a}} \geq 0$, then we call f a *(holomorphic) modular form* (of weight k with respect to υ and Γ). The space of all modular forms of weight k with respect to υ and Γ is denoted by $\mathcal{M}_k(\Gamma, \upsilon)$.

Similarly, if one may restrict the sum in (3.41) to $m + \eta_{\mathfrak{a}} > 0$ for every cusp $\mathfrak{a}$ of Γ, then we call f *(holomorphic) cusp form* (of weight k with respect to υ and Γ). The space of all cusp forms of weight k with respect to υ and Γ is denoted by $\mathcal{S}_k(\Gamma, \upsilon)$.

Proposition 3.7.1. *The spaces $\mathcal{M}_k(\Gamma, \upsilon)$ and $\mathcal{S}_k(\Gamma, \upsilon)$ are finite-dimensional $\mathbb{C}$-vector spaces with*

$$\dim_{\mathbb{C}} \mathcal{S}_k(\Gamma, \upsilon) \leq \dim_{\mathbb{C}} \mathcal{M}_k(\Gamma, \upsilon) \ll (k+1)(\mathrm{vol}\, \mathcal{F}_\Gamma + 1).$$

Proof. See [Iwa97, Section 2.7]. □

Interestingly, there is a nice characterisation of holomorphic modular forms in terms of Maass forms of a special kind. We have the following lemma.

Lemma 3.7.2. *Let $f(z)$ be a holomorphic modular form of weight k with respect to υ and Γ. Then, $y^{\frac{k}{2}} f(z)$ is a Maass form of weight $\kappa = k$ with respect to υ and Γ with eigenvalue $\frac{k}{2}(1-\frac{k}{2})$. This map further constitutes an isomorphism of vector spaces. In addition, $y^{\frac{k}{2}} f(z)$ is a cuspidal Maass form if and only if $f(z)$ is a holomorphic cusp form.*

Proof. This is just a sharpening of Lemma 3.5.2. The vanishing of the negative Fourier coefficients follows from Lemma 3.4.3, equations (3.42),(3.43), together with $\Lambda_\kappa f = 0$. □

Corollary 3.7.3. *For $k < 0$, we have $\mathcal{M}_k(\Gamma, \upsilon) = \{0\}$ and $\mathcal{M}_0(\Gamma, \upsilon) \subseteq \mathbb{C}$.*

Proof. Let $f \in \mathcal{M}_k(\Gamma, \upsilon)$. Then, by the Lemmata 3.7.2, 3.5.2, and 3.5.1, we have

$$k \| y^{\frac{k}{2}} f(z) \|_2^2 = \| K_k y^{\frac{k}{2}} f(z) \|_2^2.$$

The first statement follows. For the second statement, we see that this equality implies $K_0 f(z) = 0$. Thus, $f(z)$ is also an antiholomorphic function and hence constant. □

By comparing the Fourier coefficients of a modular form $f(z)$ (3.41) to the ones of $y^{\frac{k}{2}} f(z)$ (3.15) and using (A.4), we find that

$$\begin{aligned} \psi_f(\mathfrak{a}, m) &= (4\pi(m+\eta_\mathfrak{a}))^{\frac{k}{2}} \cdot \rho_{y^{\frac{k}{2}} f}(\mathfrak{a}, m), \quad m+\eta_\mathfrak{a} > 0, \\ \psi_f(\mathfrak{a}, m) &= \rho_{y^{\frac{k}{2}} f}(\mathfrak{a}, m), \quad m+\eta_\mathfrak{a} < 0, \end{aligned} \tag{3.42}$$

and for $\mathfrak{a}$ singular

$$\psi_f(\mathfrak{a}, 0) = \rho_{y^{\frac{k}{2}} f}(\mathfrak{a}, 0), \quad (it_{y^{\frac{k}{2}} f} = \tfrac{k}{2} - \tfrac{1}{2}). \tag{3.43}$$

It should come as no surprise that we can turn $\mathcal{S}_k(\Gamma, \upsilon)$, respectively $\mathcal{M}_k(\Gamma, \upsilon)$ for $0 \leq k < 1$, into a Hilbert space, where the inner product is given by

$$\langle f, g \rangle = \int_{\mathcal{F}_\Gamma} f(z) \overline{g(z)} y^k \frac{dxdy}{y^2}. \tag{3.44}$$

We let $\mathcal{B}_k^h(\Gamma, \upsilon)$ denote an orthonormal basis of this space. Inside the space of modular forms, we may once more find Poincaré series. For $m + \eta_\mathfrak{a} > 0$, they are defined as follows:

$$\mathcal{P}_{\mathfrak{a},m}^{\upsilon,k}(z) = (4\pi(m+\eta_\mathfrak{a}))^{k-1} y^{-\frac{k}{2}} \mathcal{U}_{\mathfrak{a},m}^{\upsilon,k}(z, \tfrac{k}{2}). \tag{3.45}$$

This is well-defined for $k > 2$. For $k \leq 2$, we need to make use of the analytic continuation of $\mathcal{U}^{v,k}_{\mathfrak{a},m}$, which works fine, except in the case $k = 1$, where one would need to go to the residual function instead. However, we shall only require this when $k \geq 2$. From Proposition (3.6.2), Corollary 3.6.5, and (3.42), we immediately recover the following proposition.

Proposition 3.7.4. *Let $k \geq 2$, $m + \eta_{\mathfrak{a}} > 0$ and $h \in \mathcal{S}_k(\Gamma, v)$. Then, we have*

$$\langle \mathcal{P}^{v,k}_{\mathfrak{a},m}, h\rangle = \overline{\psi_h(\mathfrak{a}, m)} \cdot \Gamma(k-1). \tag{3.46}$$

We shall also compute the Fourier expansion of the Poincaré series.

Proposition 3.7.5. *Let $k \geq 2$ and $m + \eta_{\mathfrak{a}} > 0$. Then, we have*

$$\begin{aligned}(\mathcal{P}^{v,k}_{\mathfrak{a},m}|_k\sigma_{\mathfrak{b}})(z) = \delta_{\mathfrak{a},\mathfrak{b}}(4\pi(m+\eta_{\mathfrak{a}}))^{k-1}e((m+\eta_{\mathfrak{a}})z) + 2\pi \sum_{n+\eta_{\mathfrak{b}}>0}\left(4\pi\sqrt{(m+\eta_{\mathfrak{a}})(n+\eta_{\mathfrak{b}})}\right)^{k-1} \\ \times \sum_{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}} \frac{S^{v,k}_{\mathfrak{a},\mathfrak{b}}(m,n;c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{(m+\eta_{\mathfrak{a}})(n+\eta_{\mathfrak{b}})}}{c}\right) e((n+\eta_{\mathfrak{b}})z),\end{aligned} \tag{3.47}$$

where for $k = 2$ the sum over $c \in \mathcal{C}_{\mathfrak{a},\mathfrak{b}}$ is to be interpret as the limit[d]

$$\lim_{\sigma\to 1^+} \sum_{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}} \frac{S^{v,k}_{\mathfrak{a},\mathfrak{b}}(m,n;c)}{c^{2\sigma-1}} J_{k-1}\left(\frac{4\pi\sqrt{(m+\eta_{\mathfrak{a}})(n+\eta_{\mathfrak{b}})}}{c}\right).$$

Proof. We have $(\mathcal{P}^{v,k}_{\mathfrak{a},m}|_k\sigma_{\mathfrak{b}})(z) = (4\pi(m+\eta_{\mathfrak{a}}))^{k-1}y^{-\frac{k}{2}}(\mathcal{U}^{v,k}_{\mathfrak{a},m}|_\kappa\sigma_{\mathfrak{b}})(z, \frac{k}{2})$. We shall make use of the Fourier expansion (3.16). Suppose first that $k > 2$. In this case, we are in the region of absolute convergence of the non-holomorphic Poincaré series and we may just set $s = \frac{k}{2}$. We shall evaluate $B^k(c, m, n, y, \frac{k}{2})$ for $k > 1$.

$$\begin{aligned}B^k(c,m,n,y,\tfrac{k}{2}) &= \int_{-\infty}^{\infty} e\left(-\frac{m}{ic^2(y-it)} - nt\right)(y-it)^{-k}dt \\ &= -ie(niy)\int_{(y)} \exp\left(-\frac{2\pi m}{c^2 t} + 2\pi n t\right) t^{-k}dt.\end{aligned}$$

d Note that this limit exists due the Taylor expansion of the Bessel function around 0 and the fact that for $m + \eta_{\mathfrak{a}}, n + \eta_{\mathfrak{b}} > 0$ the Kloosterman zeta function $\mathcal{Z}^{v}_{\mathfrak{a},\mathfrak{b}}(m, n; s)$ has no pole at $s = 1$.

For $n \leq 0$, we may shift the contour all the way to the right and find that the integral is arbitrarily small. Therefore, it must equal 0. For $n > 0$, we transform the contour into a lock-hole contour

$$\begin{aligned} B^k(c,m,n,y,\tfrac{k}{2}) &= -ie(niy) \int_{-\infty}^{(0+)} \exp\left(-\frac{2\pi m}{c^2 t} + 2\pi n t\right) t^{-k} dt \\ &= -ie(niy)(2\pi n)^{k-1} \int_{-\infty}^{(0+)} \exp\left(u - \frac{16mn}{c^2}\frac{1}{4u}\right) u^{-(k-1)} \frac{du}{u} \\ &= 2\pi e(niy) \left(\frac{n}{m}\right)^{\frac{k-1}{2}} c^{k-1} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right), \end{aligned}$$

where we made use of (A.6). Inserting this equality yields (3.47). For $k = 2$, we may proceed similarly. In this case, we may take the limit $s \to \frac{k}{2}^+ = 1^+$ of each Fourier coefficient. We find that $B^k(c,m,n,y,s)$ is an analytic function for $\mathrm{Re}(s) > \frac{1}{2}$. By using the Taylor expansion, we find

$$\begin{aligned} B^k(c,m,n,y,s) = B^k(c,m,n,y,\tfrac{k}{2}) + (s-1)\frac{d}{ds}\Big|_{s=\frac{k}{2}} B^k(c,m,n,y,s) \\ + \frac{(s-1)^2}{2}\frac{d^2}{ds^2}\Big|_{s=\xi} B^k(c,m,n,y,s), \end{aligned}$$

for some $\xi \in [\frac{k}{2}, s]$. We have

$$\begin{aligned} &\frac{d^2}{ds^2}\Big|_{s=\xi} B^k(c,m,n,y,s) \\ &\quad = \int_{-\infty}^{\infty} e\left(-\frac{m}{c^2(t+iy)} - nt\right)(y-it)^{-\frac{k}{2}-\xi}(y+it)^{-\frac{k}{2}+\xi}\log(t^2+y^2)^2 dt \\ &\quad = O_{m,n,y}(1). \end{aligned}$$

It is crucial that the error is independent of c. By inserting this expansion, we find that the first term gives us what we claim. We need to show that the other terms limit to 0. From (3.13), we conclude that

$$\sum_{c \in \mathcal{C}_{\mathfrak{a},\mathfrak{b}}} \frac{|S^{v,k}_{\mathfrak{a},\mathfrak{b}}(m,n;c)|}{c^{2s}} = O_{\mathfrak{a},\mathfrak{b}}\left(\frac{1}{s-1}\right),$$

for $s \in \mathbb{R}$ approaching 1^+. Thus, the last term also limits to 0. For the second term, we have

$$\begin{aligned} \frac{d}{ds}\Big|_{s=\frac{k}{2}} B^k(c,m,n,y,s) &= \int_{-\infty}^{\infty} e\left(-\frac{m}{c^2(t+iy)} - nt\right)(y-it)^{-k}\log(t^2+y^2)dt \\ &= \int_{-\infty}^{\infty} e\left(-nt\right)(y-it)^{-k}\log(t^2+y^2)dt + O_{m,y}\left(\frac{1}{c^2}\right), \end{aligned}$$

from which it follows that the second term also limits to 0, since for $m+\eta_{\mathfrak{a}}, n+\eta_{\mathfrak{b}} > 0$, the Kloosterman zeta function $\mathcal{Z}^{\upsilon,k}_{\mathfrak{a},\mathfrak{b}}(m,n;s)$ admits an analytic continuation to some half-plane $\mathrm{Re}(s) > 1-\delta$, for some small $\delta > 0$. □

Theorem 3.7.6 (Petersson trace formula). *Let υ be a multiplier system of weight $k \geq 2$ for the group Γ and $\mathcal{B}^h_k(\Gamma,\upsilon)$ be an orthonormal basis of $\mathcal{S}_k(\Gamma,\upsilon)$. Then, we have the identity*

$$\frac{\Gamma(k-1)}{(4\pi\sqrt{(m+\eta_{\mathfrak{a}})(n+\eta_{\mathfrak{b}})})^{k-1}} \sum_{f\in\mathcal{B}^h_k(\Gamma,\upsilon)} \overline{\psi_f(\mathfrak{a},m)}\psi_f(\mathfrak{b},n)$$
$$= \delta_{\mathfrak{a},\mathfrak{b}}\delta_{m,n} + 2\pi \sum_{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}} \frac{S^{\upsilon,k}_{\mathfrak{a},\mathfrak{b}}(m,n;c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{(m+\eta_{\mathfrak{a}})(n+\eta_{\mathfrak{b}})}}{c}\right) \tag{3.48}$$

for $m+\eta_{\mathfrak{a}}, n+\eta_{\mathfrak{b}} > 0$, where for $k=2$ the sum over $c \in \mathcal{C}_{\mathfrak{a},\mathfrak{b}}$ is to be interpret as the limit

$$\lim_{\sigma\to 1^+} \sum_{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}} \frac{S^{\upsilon,k}_{\mathfrak{a},\mathfrak{b}}(m,n;c)}{c^{2\sigma-1}} J_{k-1}\left(\frac{4\pi\sqrt{(m+\eta_{\mathfrak{a}})(n+\eta_{\mathfrak{b}})}}{c}\right).$$

Proof. We shall evaluate $\langle \mathcal{P}^{\upsilon,k}_{\mathfrak{a},m}, \mathcal{P}^{\upsilon,k}_{\mathfrak{b},n}\rangle$ in two ways. On the one hand, we have by using the Fourier expansion of the Poincaré series (3.47) together with (3.46) that

$$\langle \mathcal{P}^{\upsilon,k}_{\mathfrak{a},m}, \mathcal{P}^{\upsilon,k}_{\mathfrak{b},n}\rangle = \Gamma(k-1)\left(4\pi\sqrt{(m+\eta_{\mathfrak{a}})(n+\eta_{\mathfrak{b}})}\right)^{k-1}$$
$$\times\left[\delta_{\mathfrak{a},\mathfrak{b}}\delta_{m,n} + 2\pi \sum_{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}} \frac{S^{\upsilon,k}_{\mathfrak{a},\mathfrak{b}}(m,n;c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{(m+\eta_{\mathfrak{a}})(n+\eta_{\mathfrak{b}})}}{c}\right)\right].$$

On the other hand, we have

$$\mathcal{P}^{\upsilon,k}_{\mathfrak{a},m}(z) = \sum_{f\in\mathcal{B}_k(\Gamma,\upsilon)} \langle \mathcal{P}^{\upsilon,k}_{\mathfrak{a},m}, f\rangle f(z) = \Gamma(k-1) \sum_{f\in\mathcal{B}_k(\Gamma,\upsilon)} \overline{\psi_f(\mathfrak{a},m)} f(z)$$

and thus

$$\langle \mathcal{P}^{\upsilon,k}_{\mathfrak{a},m}, \mathcal{P}^{\upsilon,k}_{\mathfrak{b},n}\rangle = \Gamma(k-1)^2 \sum_{f\in\mathcal{B}_k(\Gamma,\upsilon)} \overline{\psi_f(\mathfrak{a},m)}\psi_f(\mathfrak{b},n).$$

□

3.8 ANTIHOLOMORPHIC MODULAR FORMS

The situation for antiholomorphic modular forms is essentially the same as for holomorphic modular forms, except for the fact that everything is complex-conjugated.

Definition 3.8.1. Let $f : \mathbb{H} \to \mathbb{C}$ be an antiholomorphic function. Then, f is called an *antiholomorphic modular form* with respect to Γ and υ of weight k if and only if $\overline{f}$ is a modular form with respect to Γ and $\overline{\upsilon}$ of weight k. The space of antiholomorphic forms with respect to Γ and υ of weight k is denoted by $\overline{\mathcal{M}}_k(\Gamma, \upsilon) = \overline{\mathcal{M}_k(\Gamma, \overline{\upsilon})}$. Similarly, f is cuspidal if and only if $\overline{f}$ is and the space of antiholomorphic cusp forms with respect to Γ and υ of weight k is denoted by $\overline{\mathcal{S}}_k(\Gamma, \upsilon) = \overline{\mathcal{S}_k(\Gamma, \overline{\upsilon})}$.

What this means is, that the holomorphic counterpart definition of being a modular function carries over to the antiholomorphic setting, if we define the slash operators for antiholomorphic functions as follows:

$$(f|_k\gamma)(z) = j(\gamma, \overline{z})^{-k} f(\gamma z).$$

The Fourier expansion of an antiholomorphic modular form $f \in \overline{\mathcal{M}}_k(\Gamma, \upsilon)$ is given by

$$(f|_k\sigma_{\mathfrak{a}})(z) = \sum_{m+\eta_{\mathfrak{a}} \leq 0} \psi_f(\mathfrak{a}, m) e((m + \eta_{\mathfrak{a}})\overline{z}).$$

Complex conjugation shows

$$\psi_f(\mathfrak{a}, m) = \overline{\psi_{\overline{f}}(\mathfrak{a}, -m - \delta_{\mathfrak{a}}^{ns})}.$$

The inner product on $\overline{\mathcal{S}}_k(\Gamma, \upsilon)$, respectively $\overline{\mathcal{M}}_k(\Gamma, \upsilon)$ for $k < 1$, is given by

$$\langle f, g \rangle = \int_{\mathcal{F}_\Gamma} f(z)\overline{g(z)} y^k \frac{dxdy}{y^2}.$$

We denote by $\mathcal{B}_k^a(\Gamma, \upsilon)$ an orthonormal basis of this space. We also have a to Lemma 3.7.2 equivalent lemma.

Lemma 3.8.1. *Let $f(z)$ be an antiholomorphic modular form of weight k with respect to υ and Γ. Then, $y^{\frac{k}{2}} f(z)$ is a Maass form of weight $\kappa = -k$ with respect to υ and Γ with eigenvalue $\frac{k}{2}(1 - \frac{k}{2})$. This map further constitutes an isomorphism of vector spaces. In addition, $y^{\frac{k}{2}} f(z)$ is a cuspidal Maass form if and only if $f(z)$ is an antiholomorphic cusp form.*

3.9 HECKE ALGEBRA

The theory of Hecke operators finds its origin in a paper of Mordell [Mor17] who used them to show multiplicative properties of the Fourier coefficients of the discriminant modular form. However, the theory as we know it today was largely developed by

Hecke [Hec37a, Hec37b] and Petersson [Pet39a, Pet39b, Pet40], with large contributions coming from Atkin–Lehner [AL70], who introduced the notion of newforms. We shall present here a theory of Hecke operators for arbitrary real weight, which was developed by Wohlfahrt [Woh57]. We shall see here that this theory has its difficulties, since it will turn out that most operators will be identically zero. There is one important exception to this, namely the half-integral weight modular forms. This case has been thoroughly investigated by Shimura [Shi73].

In order to define Hecke operators, we need to consider the larger group $\mathfrak{S}_\kappa$, whose elements consists of all pairs $(\gamma, u) \in \mathrm{SL}_2(\mathbb{R}) \times S^1$ and the group law is given by

$$(\gamma, u) \circ (\tau, v) = (\gamma\tau, uv\sigma_\kappa(\gamma, \tau)), \quad \forall(\gamma, u), (\tau, v) \in \mathfrak{S}_\kappa.$$

By making use of the relation (3.2), we easily see that this group law indeed turns $\mathfrak{S}_\kappa$ into a group with identity element $(I, 1)$. We can now extend the definition of the slash operators $|_\kappa$ and $|_k$ for $k \equiv \kappa \bmod(2)$ to elements of $\mathfrak{S}_\kappa$:

$$f|_\kappa(\gamma, u) = \overline{u} \cdot f|_\kappa\gamma \quad \text{and} \quad f|_k(\gamma, u) = \overline{u} \cdot f|_k\gamma.$$

We see that γ and $(\gamma, 1)$ induce the same operation and therefore we may identify them. However, note that this does not give rise to a group embedding. Given a Fuchsian group of the first kind Γ and a multiplier system v for Γ of weight κ. Then, we have a group embedding

$$\begin{aligned} {}^\star : \Gamma &\to \mathfrak{S}_\kappa, \\ \gamma &\mapsto \gamma^\star := (\gamma, v(\gamma)). \end{aligned}$$

We immediately see that a function being modular with respect to v and Γ is equivalent to $f|_\kappa\gamma^\star = f$, respectively $f|_k\gamma^\star = f$, for all $\gamma \in \Gamma$.

Definition 3.9.1. Let Γ be a Fuchsian group of the first kind and v a multiplier system of weight κ. Then, the *commensurator* of Γ, respectively $\Gamma^\star$ is the set of all $\gamma \in \mathrm{SL}_2(\mathbb{R})$, respectively $\xi \in \mathfrak{S}_\kappa$, such that $\Gamma \cap \gamma\Gamma\gamma^{-1}$ has finite index in Γ and $\gamma\Gamma\gamma^{-1}$, respectively $\xi\Gamma^\star\xi^{-1}$ has finite index in $\Gamma^\star$ and $\xi\Gamma^\star\xi^{-1}$. The commensurator of Γ, respectively $\Gamma^\star$, is denoted by Ξ_Γ, respectively $\Xi_{\Gamma^\star}$.

Elements $\xi \in \Xi_{\Gamma^\star}$ of the commensurator are of importance, since for them the double-coset quotient $\Gamma^\star \setminus \Gamma^\star\xi\Gamma^\star$ is of finite order and we may define operators $|_\kappa\Gamma^\star\xi\Gamma^\star : \mathcal{H}_\kappa(\Gamma, v) \to \mathcal{H}_\kappa(\Gamma, v)$ and $|_k\Gamma^\star\xi\Gamma^\star : \mathcal{M}_k(\Gamma, v) \to \mathcal{M}_k(\Gamma, v)$ as follows:

$$f|_\kappa\Gamma^\star\xi\Gamma^\star = \sum_{\eta \in \Gamma^\star \setminus \Gamma^\star\xi\Gamma^\star} f|_\kappa\eta, \qquad f|_k\Gamma^\star\xi\Gamma^\star = \sum_{\eta \in \Gamma^\star \setminus \Gamma^\star\xi\Gamma^\star} f|_k\eta.$$

We shall note here that $|_\kappa \Gamma^\star \xi \Gamma^\star$ commutes with Δ_κ and $|_k \Gamma^\star \xi \Gamma^\star$ preserves the space of cusp forms $\mathcal{S}_k(\Gamma, \upsilon)$. The operators are in essence our Hecke operators. However, we shall normalise them when we apply them to congruence subgroups.

Proposition 3.9.1. *Let $\tau \in \Xi_\Gamma$ and $\upsilon \in S^1$. Then, $\xi = (\tau, \upsilon) \in \Xi_{\Gamma^\star}$ if and only if the kernel of the character $t : \Gamma \cap \tau^{-1}\Gamma\tau \to S^1$ given by*

$$t(\gamma) = \frac{\upsilon^\tau(\gamma)}{\upsilon(\gamma)}, \quad \forall \gamma \in \Gamma \cap \tau^{-1}\Gamma\tau,$$

has finite index in $\Gamma \cap \tau^{-1}\Gamma\tau$, where we recall the conjugated multiplier system υ^τ from Proposition 3.2.5. In which case, the operators $|_\kappa \Gamma^\star \xi \Gamma^\star$ and $|_k \Gamma^\star \xi \Gamma^\star$ are the zero operators, unless $t(\gamma) = 1$ for all $\gamma \in \Gamma \cup \tau^{-1}\Gamma\tau$.

Proof. See [Shi73, Prop. 1.0]. □

From this proposition, we see that the multiplier system must be of special shape to allow for non-trivial operators $|_\kappa \Gamma^\star \xi \Gamma^\star$, $|_k \Gamma^\star \xi \Gamma^\star$.

Proposition 3.9.2. *Let $\xi \in \Xi_{\Gamma^\star}$ and $f, g \in \mathcal{H}_\kappa(\Gamma, \upsilon)$, or $f, g \in S_k(\Gamma, \upsilon)$. Then, we have*

$$\langle f|_\kappa \Gamma^\star \xi \Gamma^\star, g\rangle = \langle f, g|_\kappa \Gamma^\star \xi^{-1} \Gamma^\star\rangle, \text{ respectively } \langle f|_k \Gamma^\star \xi \Gamma^\star, g\rangle = \langle f, g|_k \Gamma^\star \xi^{-1} \Gamma^\star\rangle.$$

Proof. See [Shi71, Prop. 3.39]. □

It is further possible to consider the $\mathbb{C}$-module generated by double cosets $\Gamma^\star \xi \Gamma^\star$ with $\xi \in \Xi_{\Gamma^\star}$, which is denoted by $R(\Gamma^\star, \Xi_{\Gamma^\star})$ and define a multiplication law on it, which is consistent with the $|_\kappa$ operator. We refer the reader to [Shi71, Chapter 3].

From here on, we only consider a special case of the general theory. We shall consider the group $\Gamma = \Gamma_0(N)$, for some $N \in \mathbb{N}$, with trivial multiplier system of weight 0. Note that since κ is integral, we have $\mathfrak{G}_\kappa \cong \mathrm{SL}_2(\mathbb{R}) \times S^1$ as groups and we may forget about the second coordinate. It is easily verified that

$$\left\{ \gamma \in \mathrm{SL}_2(\mathbb{R}) \middle| \gamma = \frac{1}{\sqrt{\det(\tau)}}\tau \text{ for some } \tau \in \mathrm{Mat}_{2\times 2}(\mathbb{Z}) \text{ with } \det(\tau) > 0 \right\} \subseteq \Xi_{\Gamma_0(N)}.$$

We define the Hecke operators as

$$|_0 T_m = m^{-\frac{1}{2}} \cdot |_0 \Gamma_0(N) \begin{pmatrix} 1/\sqrt{m} & 0 \\ 0 & \sqrt{m} \end{pmatrix} \Gamma_0(N)$$
$$|_k T_m = m^{\frac{k}{2}-1} \cdot |_k \Gamma_0(N) \begin{pmatrix} 1/\sqrt{m} & 0 \\ 0 & \sqrt{m} \end{pmatrix} \Gamma_0(N).$$

A set of representatives of $\Gamma_0(N)\setminus\Gamma_0(N)\left(\begin{smallmatrix}1/\sqrt{m} & 0\\ 0 & \sqrt{m}\end{smallmatrix}\right)\Gamma_0(N)$ may be given by

$$\left\{\frac{1}{\sqrt{m}}\begin{pmatrix}a & b\\ 0 & d\end{pmatrix}\,\middle|\, a,d\in\mathbb{N}, ad=m, (a,N)=1, b=0,\ldots,d-1\right\}. \tag{3.49}$$

These operators commute with each other (see for example [Shi71, Prop. 3.8]). Moreover, they are multiplicative (see for example [Shi71, Chapter 3.3]), i.e. we have

$$|_0T_{mn} = |_0T_m|_0T_n = |_0T_n|_0T_m, \quad (m,n)=1,$$

and

$$|_kT_{mn} = |_kT_m|_kT_n = |_kT_n|_kT_m, \quad (m,n)=1.$$

Since we have $\Gamma_0(N)\left(\begin{smallmatrix}1/\sqrt{m} & 0\\ 0 & \sqrt{m}\end{smallmatrix}\right)\Gamma_0(N) = \Gamma_0(N)\left(\begin{smallmatrix}\sqrt{m} & 0\\ 0 & 1/\sqrt{m}\end{smallmatrix}\right)\Gamma_0(N)$ for $(m,N)=1$, we have by means of Proposition 3.9.2, that the operators $|_0T_m$ and $|_kT_m$ for $(m,N)=1$ are self-adjoint. Hence, we may simultaneously diagonalise the spaces $\mathcal{H}_0(\Gamma_0(N),1)$ and $\mathcal{S}_k(\Gamma_0(N),1)$. This is clear for the latter space as they are finite-dimensional. For the former, we may diagonalise each eigenspace $\mathcal{H}_0(\Gamma_0(N),1,\lambda)$. Recall, that this space is finite-dimensional and that the Hecke operators commute with the Laplace–Beltrami operator Δ_0.

Definition 3.9.2. A cusp form $f\in\mathcal{S}_k(\Gamma_0(N),1)$ is called a *Hecke eigenform* if it is an eigenfunction of all the Hecke operators $|_kT_m$ for $(m,N)=1$. Analogously, a Maass form $f\in\mathcal{H}_0(\Gamma_0(N),1)$ is called a *Hecke–Maass eigenform* if it is an eigenfunction of all the Hecke operators $|_0T_m$ for $(m,N)=1$.

The Hecke(–Maass) eigenforms fall into different categories, so-called oldforms and newforms. We shall summarise the results of [AL70], but first we need to introduce another operator, which increases the level.

Lemma 3.9.3. *Let A_n denote the matrix $\left(\begin{smallmatrix}\sqrt{n} & 0\\ 0 & 1/\sqrt{n}\end{smallmatrix}\right)$. Then, $|_0A_n$ defines a map from $\mathcal{H}_0(\Gamma_0(N),1)\to\mathcal{H}_0(\Gamma_0(nN),1)$ and $|_kA_n$ defines a map from $\mathcal{S}_k(\Gamma_0(N),1)\to\mathcal{S}_k(\Gamma_0(nN),1)$. Moreover $|_0A_n$, respectively $|_kA_n$, commutes with all Hecke operators $|_0T_m$, respectively $|_kT_m$, for $(m,n)=1$.*

Proof. The first part follows from $A_n\Gamma_0(nN)A_n^{-1}\subseteq\Gamma_0(N)$ and the second part follows from the fact that conjugating the coset representatives (3.49) with A_n just permutes them ($b\mapsto bn \bmod(d)$). □

Theorem 3.9.4. *The spaces $\mathcal{H}_0(\Gamma_0(N),1)$ and $S_k(\Gamma_0(N),1)$ split into an oldspace, which is generated by $\mathcal{H}_0(\Gamma_0(M),1)|_0 A_n$, respectively $\mathcal{S}_k(\Gamma_0(M),1)|_0 A_n$, where M runs over all proper divisors of N and n over all divisors of $\frac{N}{M}$, and its orthogonal complement the newspace. Each Hecke(–Maass) eigenform either falls into the oldspace, in which case we call it an oldform, or into the newspace, in which case we call it a newform. The whole of the oldspace is generated by oldforms and every oldform is in the span of forms $f|_0 A_n$, respectively $f|_k A_n$, where f is a fixed newform of level M for some proper divisor of N and n runs over all the divisors of $\frac{N}{M}$. Each newform f is an eigenfunction of all the Hecke operators (including the ones dividing the level). Moreover, if we denote by $\lambda_f(m)$ the eigenvalue of $|_0 T_m$, respectively $|_k T_m$, then we have*

$$\begin{aligned}\sum_{n\geq 1}\rho_f(\infty,\pm n)n^{\frac{1}{2}-s} &= \rho_f(\infty,\pm 1)\sum_{n\geq 1}\lambda_f(n)n^{-s}\\ &= \rho_f(\infty,\pm 1)\prod_{p|N}\left(1-\lambda_f(p)p^{-s}\right)^{-1}\prod_{p\nmid N}\left(1-\lambda_f(p)p^{-s}+p^{-2s}\right)^{-1},\end{aligned}$$

respectively

$$\begin{aligned}\sum_{n\geq 1}\psi_f(\infty,n)n^{-s} &= \psi_f(\infty,1)\sum_{n\geq 1}\lambda_f(n)n^{-s}\\ &= \psi_f(\infty,1)\prod_{p|N}\left(1-\lambda_f(p)p^{-s}\right)^{-1}\prod_{p\nmid N}\left(1-\lambda_f(n)p^{-s}+p^{k-1-2s}\right)^{-1},\end{aligned}$$

formally. In particular, we have $\psi_f(\infty,1)\neq 0$.

Proof. See [AL70]. □

The size of the eigenvalues of the Hecke operators, or more generally the Fourier coefficients of cusp forms, have remained a mystery for a long time. Many different approaches have been fruitful for various cases. We refer to a survey article of Selberg [Sel65]. More recently, tools of l-adic cohomology and functoriality of symmetric powers of representations have entered the picture and they remain the most successful approaches when it comes to arithmetic groups. We shall record these bounds here. If f is a Maass newform, then we have by the works of Kim–Sarnak [Kim03] $\lambda_f(n) \ll_\epsilon n^{\theta+\epsilon}$, where $\theta = \frac{7}{64}$. In the case where f is a holomorphic newform, we have $\lambda_f(n) \ll_\epsilon n^{\frac{k-1}{2}+\epsilon}$ by the works of Deligne [Del71, Del74] and Deligne–Serre [DS74].

In due course, we shall need an orthonormal basis of Hecke(–Maass) eigenforms of the space $\mathcal{H}_0(\Gamma_0(N),1)$, respectively $\mathcal{S}_k(\Gamma_0(N),1)$. The basis we present here was computed by Blomer–Milićević [BM15b][e]. For a newform h of level $M|N$, we define the arithmetic functions

$$R_h(c) = \sum_{b|c} \frac{\mu(b)\lambda_h(b)^2}{b} \left(\sum_{d|b} \frac{\chi_0(d)}{d} \right)^{-2}, \quad A(c) = \sum_{b|c} \frac{\mu(b)\chi_0(b)^2}{b^2}, \quad B(c) = \sum_{b|c} \frac{\mu(b)^2\chi_0(b)}{b},$$

where χ_0 is the trivial character modulo M, and the multiplicative function $\mu_h(c)$ is defined by the equation

$$\left(\sum_{c\geq 1} \frac{\lambda_h(c)}{c^s} \right)^{-1} = \sum_{c\geq 1} \frac{\mu_h(c)}{c^s}.$$

For $l|d$ define

$$\xi'_d(l) = \frac{\mu(d/l)\lambda_h(d/l)}{r_h(d)^{\frac{1}{2}}(d/l)^{\frac{1}{2}}B(d/l)}, \quad \xi''_d(l) = \frac{\mu_h(d/l)}{r_h(d)^{\frac{1}{2}}(d/l)^{\frac{1}{2}}A(d)^{\frac{1}{2}}}.$$

Let us write $d = d_1 d_2$ with d_1 square-free, d_2 square-full, and $(d_1, d_2) = 1$. Then, for $l|d$ define

$$\xi_d(l) = \xi'_{d_1}((d_1, l))\xi''_{d_2}((d_2, l)) \ll_\epsilon d^\epsilon. \tag{3.50}$$

Then, an orthonormal basis of $\mathcal{H}_0(\Gamma_0(N), 1)$ is given by

$$\bigcup_{M|N} \bigcup_{\substack{h \text{ new} \\ \text{of level } M}} \left\{ h^d(z) = \sum_{l|d} \xi_d(l) \cdot h|_0 A_l \,\middle|\, d|\tfrac{N}{M} \right\} \tag{3.51}$$

and an orthonormal basis of $\mathcal{S}_k(\Gamma_0(N), 1)$ is given by

$$\bigcup_{M|N} \bigcup_{\substack{f \text{ new} \\ \text{of level } M}} \left\{ f^d(z) = \sum_{l|d} \xi_d(l) \cdot f|_k A_l \,\middle|\, d|\tfrac{N}{M} \right\}. \tag{3.52}$$

We shall record here a bound for the Fourier coefficients of these orthonormal bases. Let h be an L^2-normalised Maass newform of level $M|N$ and $d|\frac{N}{M}$. Then, we have

$$\begin{aligned} \sqrt{n}\rho_{h^d}(\infty, n) &= \sum_{l|(d,n)} \sqrt{l}\xi_d(l)\lambda_h\left(\frac{n}{l}\right)\rho_h(\infty, 1) \\ &\ll_\epsilon (nN)^\epsilon n^\theta |\rho_h(\infty, 1)| \sum_{l|(d,n)} l^{\frac{1}{2}-\theta} \\ &\ll_\epsilon (nN)^\epsilon n^\theta \left(\frac{N}{M}\right)^{\frac{1}{2}} |\rho_h(\infty, 1)|, \end{aligned} \tag{3.53}$$

where we have made use of (3.50) and $\lambda_h(n) \ll_\epsilon n^{\theta+\epsilon}$. Since h is new of level N, but normalised with respect to the inner product of level N (3.30), we further have

$$|\rho_h(\infty, 1)| \ll_\epsilon (N(1 + |t_h|))^\epsilon \left(\frac{\cosh(\pi t_h)}{N}\right)^{\frac{1}{2}}, \tag{3.54}$$

e Corrections can be found at http://www.uni-math.gwdg.de/blomer/corrections.pdf.

which is due to Hoffstein–Lockhart [HL94]. If f is an L^2-normalised holomorphic newform of level $M|N$, weight k, and $d|\frac{N}{M}$, then we have

$$\begin{aligned}\psi_{f^d}(\infty, n) &= \sum_{l|(d,n)} \xi_d(l) l^{\frac{k}{2}} \lambda_f\left(\frac{n}{l}\right) \psi_f(\infty, 1) \\ &\ll_\epsilon (nN)^\epsilon n^{\frac{k-1}{2}} |\psi_h(\infty, 1)| \sum_{l|(d,n)} l^{\frac{1}{2}} \\ &\ll_\epsilon (nN)^\epsilon n^{\frac{k-1}{2}} \left(\frac{N}{M}\right)^{\frac{1}{2}} |\psi_h(\infty, 1)|,\end{aligned} \tag{3.55}$$

where we have made use of the Deligne bound as well as (3.50). We further have the bound

$$|\psi_h(\infty, 1)| \ll_\epsilon \frac{(4\pi)^{\frac{k-1}{2}}}{N^{\frac{1}{2}} \Gamma(k)^{\frac{1}{2}}} (kN)^\epsilon, \tag{3.56}$$

when h is new of level r, but normalised with respect to (3.44); see for example [Mic07, pp. 41,42].

3.10 THE KUZNETSOV TRACE FORMULA

As previously mentioned, we shall derive the Kuznetsov trace formula from the pre-trace formula and complete it with the Petersson trace formula. We shall restrict ourselves to $\kappa \in [0, 2[$. The case $m + \eta_{\mathfrak{a}}, n + \eta_{\mathfrak{b}} > 0$ with $\mathfrak{a} = \mathfrak{b}$ has already been worked out by Proskurin [Pro05]. One may easily adopt Proskurin's method to account for the general case by using Proposition 3.6.8 and Theorem 3.7.6 instead of the pre-trace formula given there. One arrives at the following theorem.

Theorem 3.10.1. *Let Γ be a Fuchsian group of the first kind, υ a multiplier system for Γ of weight $\kappa \in [0, 2[$, and $\mathfrak{a}, \mathfrak{b}$ two cusps of Γ. Let $\phi : [0, \infty[\to \mathbb{C}$ be a function with continuous derivatives up to third order satisfying*

$$\phi(0) = \phi'(0) = 0, \quad \phi(x) \ll (x+1)^{-1-\delta}, \quad \phi'(x), \phi''(x), \phi'''(x) \ll (x+1)^{-2-\delta},$$

for some $\delta > 0$. Then, for $m + \eta_{\mathfrak{a}}, n + \eta_{\mathfrak{b}} > 0$, we have[f]

$$\sum_{c \in \mathcal{C}_{\mathfrak{a},\mathfrak{b}}} \frac{S^{\upsilon,\kappa}(m, n; c)}{c} \phi\left(\frac{4\pi\sqrt{(m+\eta_{\mathfrak{a}})(n+\eta_{\mathfrak{b}})}}{c}\right) = \mathcal{H}^\kappa_\upsilon(m, n; \phi) + \mathcal{M}^\kappa_\upsilon(m, n; \phi) + \mathcal{E}^\kappa_\upsilon(m, n; \phi),$$

f We adopt the convention that if a Fourier coefficient is 0, then that summand is zero. This is only needed when $\kappa = 0$.

where

$$\mathcal{H}^{\kappa}_{\upsilon}(m,n;\phi)=\frac{1}{\pi}\sum_{\substack{k\equiv\kappa \bmod (2)\\ k>0}}\sum_{h\in\mathcal{B}^h_k(\Gamma,\upsilon)}\frac{(-1)^{\frac{k-\kappa}{2}}\Gamma(k)}{(4\pi\sqrt{(m+\eta_{\mathfrak{a}})(n+\eta_{\mathfrak{b}})})^{k-1}}\overline{\psi_h(\mathfrak{a},m)}\psi_h(\mathfrak{b},n)\widetilde{\phi}(k-1),$$

$$\mathcal{M}^{\kappa}_{\upsilon}(m,n;\phi)=4\sum_{\substack{h\in\mathcal{B}_\kappa(\Gamma,\upsilon)\\ \lambda_h>\frac{\kappa}{2}(1-\frac{\kappa}{2})}}\frac{\sqrt{(m+\eta_{\mathfrak{a}})(n+\eta_{\mathfrak{b}})}}{\cosh(\pi t_h)}\overline{\rho_h(\mathfrak{a},m)}\rho_h(\mathfrak{b},n)\widehat{\phi}(t_h,\kappa),$$

$$\mathcal{E}^{\kappa}_{\upsilon}(m,n;\phi)=\sum_{\mathfrak{c}\ \textit{sing.}}\int_{-\infty}^{\infty}\left(\frac{n+\eta_{\mathfrak{b}}}{m+\eta_{\mathfrak{a}}}\right)^{ir}\frac{\overline{\mathcal{Z}^{\upsilon,\kappa}_{\mathfrak{c},\mathfrak{a}}(0,m;\frac{1}{2}+ir)}}{\Gamma(\frac{1}{2}+\frac{\kappa}{2}-ir)}\frac{\mathcal{Z}^{\upsilon,\kappa}_{\mathfrak{c},\mathfrak{b}}(0,n;\frac{1}{2}+ir)}{\Gamma(\frac{1}{2}+\frac{\kappa}{2}+ir)}\frac{\widehat{\phi}(r,\kappa)dr}{\cosh(\pi r)}.$$

Here, the transforms are given by

$$\begin{aligned}\widetilde{\phi}(t)&=\int_0^\infty J_t(x)\phi(x)\frac{dx}{x},\\ \widehat{\phi}(t,\kappa)&=i\pi^2\frac{\displaystyle\int_0^\infty\Big[\cos\left(\pi(\tfrac{\kappa}{2}+it)\right)J_{2it}(x)-\cos\left(\pi(\tfrac{\kappa}{2}-it)\right)J_{-2it}(x)\Big]\phi(x)\frac{dx}{x}}{\sinh(\pi t)\left(\cosh(2\pi t)+\cos(\pi\kappa)\right)\Gamma(\frac{1}{2}-\frac{\kappa}{2}+it)\Gamma(\frac{1}{2}-\frac{\kappa}{2}-it)}.\end{aligned} \tag{3.57}$$

The observant eye will notice that we have excluded the very bottom of the spectrum and included it in the holomorphic contribution. We shall show that this was a valid manœuvre.

Proof. We have $\lambda_h=\frac{\kappa}{2}(1-\frac{\kappa}{2})\Leftrightarrow t_h=\pm i(\frac{\kappa}{2}-\frac{1}{2})$ and

$$\begin{aligned}\widehat{\phi}\left(\pm i\left(\tfrac{\kappa}{2}-\tfrac{1}{2}\right),\kappa\right)&=i\pi^2\frac{-\displaystyle\int_0^\infty\cos\left(\pi\left(\kappa-\tfrac{1}{2}\right)\right)J_{\kappa-1}(x)\phi(x)\frac{dx}{x}}{i\sin\left(\pi\left(\frac{\kappa}{2}-\frac{1}{2}\right)\right)(-2\pi)\sin\left(\pi\left(\kappa-1\right)\right)\Gamma(1-\kappa)\Gamma(1)}\\&=\frac{\pi}{2}\frac{\sin\left(\kappa\pi\right)}{\cos\left(\frac{\kappa}{2}\pi\right)\sin\left(\pi(1-\kappa)\right)\Gamma(1-\kappa)}\widetilde{\phi}(\kappa-1)\\&=\sin\left(\tfrac{\kappa}{2}\pi\right)\Gamma(\kappa)\widetilde{\phi}(\kappa-1).\end{aligned}$$

By invoking Lemma 3.5.2, we further have that $y^{-\frac{\kappa}{2}}h$ is holomorphic and the collection of these functions form an orthonormal basis of $\mathcal{S}_\kappa(\Gamma,\upsilon)$, respectively $\mathcal{M}_\kappa(\Gamma,\upsilon)$ if $\kappa<1$. Hence, by (3.42) we have

$$\begin{aligned}&4\sum_{\substack{h\in\mathcal{B}_\kappa(\Gamma,\upsilon)\\ \lambda_h=\frac{\kappa}{2}(1-\frac{\kappa}{2})}}\frac{\sqrt{(m+\eta_{\mathfrak{a}})(n+\eta_{\mathfrak{b}})}}{\cosh(\pi t_h)}\overline{\rho_h(\mathfrak{a},m)}\rho_h(\mathfrak{b},n)\widehat{\phi}(t_h)\\&\qquad=\frac{1}{\pi}\sum_{h\in\mathcal{B}^h_\kappa(\Gamma,\upsilon)}\frac{\Gamma(\kappa)}{\left(4\pi\sqrt{(m+\eta_{\mathfrak{a}})(n+\eta_{\mathfrak{b}})}\right)^{k-1}}\overline{\psi_h(\mathfrak{a},m)}\psi_h(\mathfrak{b},n)\widetilde{\phi}(1-\kappa).\end{aligned}$$

□

The above theorem is sufficient to spectrally expand sums of Kloosterman sums when $m+\eta_{\mathfrak{a}}$ and $n+\eta_{\mathfrak{b}}$ have the same sign. However one might be interested in expressing the negative Fourier coefficients geometrically. Unfortunately, Proskurin did not consider the case $\kappa \in]-2,0[$, so the case $m+\eta_{\mathfrak{a}}, n+\eta_{\mathfrak{b}} < 0$ does not simply follow from complex conjugation, but requires some work.

Theorem 3.10.2. *Let Γ be a Fuchsian group of the first kind, υ a multiplier system for Γ of weight $\kappa \in [0,2[$, and $\mathfrak{a},\mathfrak{b}$ two cusps of Γ. Let $\phi : [0,\infty[\to \mathbb{C}$ be a function with continuous derivatives up to third order satisfying*

$$\phi(0)=\phi'(0)=0, \quad \phi(x) \ll (x+1)^{-1-\delta}, \quad \phi'(x),\phi''(x),\phi'''(x) \ll (x+1)^{-2-\delta},$$

for some $\delta>0$. Then, for $m+\eta_{\mathfrak{a}}, n+\eta_{\mathfrak{b}} < 0$, we have

$$\sum_{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}} \frac{S^{\upsilon,\kappa}(m,n;c)}{c}\phi\left(\frac{4\pi\sqrt{(m+\eta_{\mathfrak{a}})(n+\eta_{\mathfrak{b}})}}{c}\right) = \mathcal{A}^{\kappa}_{\upsilon}(m,n;\phi)+\mathcal{M}^{\kappa}_{\upsilon}(m,n;\phi)+\mathcal{E}^{\kappa}_{\upsilon}(m,n;\phi),$$

where

$$\mathcal{A}^{\kappa}_{\upsilon}(m,n;\phi)=\frac{1}{\pi}\sum_{\substack{k\equiv-\kappa \bmod(2)\\ k>0}}\sum_{h\in\mathcal{B}^{a}_{k}(\Gamma,\upsilon)}\frac{(-1)^{\frac{k+\kappa}{2}}\Gamma(k)}{(4\pi\sqrt{(m+\eta_{\mathfrak{a}})(n+\eta_{\mathfrak{b}})})^{k-1}}\overline{\psi_h(\mathfrak{a},m)}\psi_h(\mathfrak{b},n)\widetilde{\phi}(k-1),$$

$$\mathcal{M}^{\kappa}_{\upsilon}(m,n;\phi)=4\sum_{\substack{h\in\mathcal{B}_{\kappa}(\Gamma,\upsilon)\\ \lambda_h>\frac{\kappa}{2}(1-\frac{\kappa}{2})}}\frac{\sqrt{(m+\eta_{\mathfrak{a}})(n+\eta_{\mathfrak{b}})}}{\cosh(\pi t_h)}\overline{\rho_h(\mathfrak{a},m)}\rho_h(\mathfrak{b},n)\widehat{\phi}(t_h,-\kappa),$$

$$\mathcal{E}^{\kappa}_{\upsilon}(m,n;\phi)=\sum_{\mathfrak{c}\ \mathrm{sing.}}\int_{-\infty}^{\infty}\left(\frac{n+\eta_{\mathfrak{b}}}{m+\eta_{\mathfrak{a}}}\right)^{ir}\frac{\overline{\mathcal{Z}^{\upsilon,\kappa}_{\mathfrak{c},\mathfrak{a}}(0,m;\frac{1}{2}+ir)}}{\Gamma(\frac{1}{2}-\frac{\kappa}{2}-ir)}\frac{\mathcal{Z}^{\upsilon,\kappa}_{\mathfrak{c},\mathfrak{b}}(0,n;\frac{1}{2}+ir)}{\Gamma(\frac{1}{2}-\frac{\kappa}{2}+ir)}\frac{\widehat{\phi}(r,-\kappa)dr}{\cosh(\pi r)},$$

and the transforms are given as in (3.57).

Proof. The case $\kappa=0$ follows directly from Theorem 3.10.1 and complex conjugation. Now, we consider the case $\kappa\in]0,2[$. Our starting point is Theorem 3.10.1 with the multiplier system $\overline{\upsilon}$ of weight $2-\kappa$ with entries $-m-\delta^{ns}_{\mathfrak{a}}, -n-\delta^{ns}_{\mathfrak{b}}$ in the Kloosterman sum. The transform is given by

$$\begin{aligned}
\widehat{\phi}(t,2-\kappa) =& i\pi^2\frac{\int_0^\infty\left[\cos\left(\pi(\frac{2-\kappa}{2}+it)\right)J_{2it}(x)-\cos\left(\pi(\frac{2-\kappa}{2}-it)\right)J_{-2it}(x)\right]\phi(x)\frac{dx}{x}}{\sinh(\pi t)\left(\cosh(2\pi t)+\cos(\pi(2-\kappa))\right)\Gamma(\frac{1}{2}-\frac{2-\kappa}{2}+it)\Gamma(\frac{1}{2}-\frac{2-\kappa}{2}-it)}\\
=&-i\pi^2\frac{\int_0^\infty\left[\cos\left(\pi(\frac{-\kappa}{2}+it)\right)J_{2it}(x)-\cos\left(\pi(\frac{-\kappa}{2}-it)\right)J_{-2it}(x)\right]\phi(x)\frac{dx}{x}}{\sinh(\pi t)\left(\cosh(2\pi t)+\cos(-\pi\kappa)\right)\Gamma(\frac{1}{2}+\frac{\kappa}{2}+it)\Gamma(\frac{1}{2}+\frac{\kappa}{2}-it)}\\
&\times\left(\frac{\kappa}{2}-\frac{1}{2}+it\right)\left(\frac{\kappa}{2}-\frac{1}{2}-it\right)\\
=&-\left(t^2+\left(\tfrac{\kappa}{2}-\tfrac{1}{2}\right)^2\right)\widehat{\phi}(t,-\kappa).
\end{aligned}$$

Now, since $\lambda_h > \frac{2-\kappa}{2}(1-\frac{2-\kappa}{2})$, we may apply the weight lowering operator $\Lambda_{2-\kappa}$ to our basis. By Lemmata 3.4.2, 3.5.1, this preserves the eigenvalue and, when restricted to a fixed eigenvalue $> \frac{2-\kappa}{2}(1-\frac{2-\kappa}{2})$, is an orthogonality preserving bijection. Furthermore, we have for $h \in \mathcal{B}_{2-\kappa}(\Gamma, \overline{\upsilon})$ with $\lambda_h > \frac{2-\kappa}{2}(1-\frac{2-\kappa}{2})$:

$$\begin{aligned}\|\Lambda_{2-\kappa}h\|^2 &= \left(\lambda_h - \tfrac{2-\kappa}{2}(1-\tfrac{2-\kappa}{2})\right)\|h\|^2 \\ &= \left(t_h^2 + \left(\tfrac{\kappa}{2}-\tfrac{1}{2}\right)^2\right)\|h\|^2\end{aligned}$$

and by Lemma 3.4.3 also

$$\begin{aligned}\rho_{\Lambda_{2-\kappa}h}(\mathfrak{a}, -m-\delta_{\mathfrak{a}}^{ns}) &= -\left(t_h^2 + \left(\tfrac{2-\kappa}{2}-\tfrac{1}{2}\right)^2\right)\rho_h(\mathfrak{a}, -m-\delta_{\mathfrak{a}}^{ns}) \\ &= -\left(t_h^2 + \left(\tfrac{\kappa}{2}-\tfrac{1}{2}\right)^2\right)\rho_h(\mathfrak{a}, -m-\delta_{\mathfrak{a}}^{ns}).\end{aligned}$$

Hence, we find by normalising that

$$\begin{aligned}&\mathcal{M}_{\overline{\upsilon}}^{2-\kappa}(-m-\delta_{\mathfrak{a}}^{ns}, -n-\delta_{\mathfrak{b}}^{ns}, \phi) \\ &\quad= -4\sum_{\substack{h\in\mathcal{B}_{-\kappa}(\Gamma,\overline{\upsilon}) \\ \lambda_h > \frac{\kappa}{2}(1-\frac{\kappa}{2})}} \frac{\sqrt{(m+\eta_{\mathfrak{a}}^{\upsilon})(n+\eta_{\mathfrak{b}}^{\upsilon})}}{\cosh(\pi t_h)}\overline{\rho_h(\mathfrak{a}, -m-\delta_{\mathfrak{a}}^{ns})}\rho_h(\mathfrak{b}, -n-\delta_{\mathfrak{b}}^{ns})\widehat{\phi}(t_h, -\kappa).\end{aligned}$$

We also find that $\mathcal{E}_{\overline{\upsilon}}^{2-\kappa}(-m-\delta_{\mathfrak{a}}^{ns}, -n-\delta_{\mathfrak{b}}^{ns}; \phi)$ is equal to

$$-\sum_{\mathfrak{c}\text{ sing.}}\int_{-\infty}^{\infty}\left(\frac{n+\eta_{\mathfrak{b}}^{\upsilon}}{m+\eta_{\mathfrak{a}}^{\upsilon}}\right)^{ir}\frac{\overline{\mathcal{Z}_{\mathfrak{c},\mathfrak{a}}^{\overline{\upsilon},-\kappa}(0, -m-\delta_{\mathfrak{a}}^{ns}; \frac{1}{2}+ir)}}{\Gamma(\frac{1}{2}-\frac{\kappa}{2}-ir)}\frac{\mathcal{Z}_{\mathfrak{c},\mathfrak{b}}^{\overline{\upsilon},-\kappa}(0, -n-\delta_{\mathfrak{b}}^{ns}; \frac{1}{2}+ir)}{\Gamma(\frac{1}{2}-\frac{\kappa}{2}+ir)}\frac{\widehat{\phi}(r, -\kappa)dr}{\cosh(\pi r)}.$$

Hence, by complex conjugating, we conclude the proof of the theorem. □

The final case, where $m+\eta_{\mathfrak{a}} > 0$ and $n+\eta_{\mathfrak{b}} < 0$, is slightly easier to prove from scratch. Nevertheless, there are quite a few convergence issues. The proof is essentially a combination of the proofs of [DI83], [Pro05], [Pro79], and [AA18].

Theorem 3.10.3. *Let Γ be a Fuchsian group of the first kind, υ a multiplier system for Γ of weight $\kappa \in]-2, 2[$, and $\mathfrak{a}, \mathfrak{b}$ two cusps of Γ. Let $\phi : [0, \infty[\to \mathbb{C}$ be a function with continuous derivatives up to third order satisfying*

$$\phi(0) = \phi'(0) = 0, \quad \phi(x), \phi'(x), \phi''(x) \ll (x+1)^{-1-\delta},$$

for some $\delta > 0$. Then, for $m+\eta_{\mathfrak{a}} > 0$, $n+\eta_{\mathfrak{b}} < 0$, we have

$$\sum_{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}}\frac{S^{\upsilon,\kappa}(m,n;c)}{c}\phi\left(\frac{4\pi\sqrt{(m+\eta_{\mathfrak{a}})|n+\eta_{\mathfrak{b}}|}}{c}\right) = \mathcal{M}_{\upsilon}^{\kappa}(m,n;\phi) + \mathcal{E}_{\upsilon}^{\kappa}(m,n;\phi),$$

where

$$\mathcal{M}_{\upsilon}^{\kappa}(m,n;\phi) = 2\pi \sum_{h\in\mathcal{B}_\kappa(\Gamma,\upsilon)} \frac{\sqrt{(m+\eta_{\mathfrak{b}})|n+\eta_{\mathfrak{b}}|}}{\cosh(\pi t_h)} \overline{\rho_h(\mathfrak{a},m)}\rho_h(\mathfrak{b},n)\check{\phi}(t_h),$$

$$\mathcal{E}_{\upsilon}^{\kappa}(m,n;\phi) = \frac{\pi}{2}\sum_{\mathfrak{c}\text{ sing.}} \int_{-\infty}^{\infty} \left|\frac{m+\eta_{\mathfrak{a}}}{n+\eta_{\mathfrak{b}}}\right|^{-ir} \frac{\overline{\mathcal{Z}_{\mathfrak{c},\mathfrak{a}}^{\upsilon,\kappa}(0,m;\frac{1}{2}+ir)}}{\Gamma(\frac{1}{2}+\frac{\kappa}{2}-ir)} \frac{\mathcal{Z}_{\mathfrak{c},\mathfrak{b}}^{\upsilon,\kappa}(0,n;\frac{1}{2}+ir)}{\Gamma(\frac{1}{2}-\frac{\kappa}{2}+ir)} \frac{\check{\phi}(r)dr}{\cosh(\pi r)},$$

and the transform is given by

$$\check{\phi}(r) = \frac{4}{\pi}\cosh(\pi r)\int_0^\infty K_{2ir}(x)\phi(x)\frac{dx}{x}.$$

Remark 3.10.4. The only place where the third derivative of ϕ is used is at zero for the Kontorovich–Lebedev inversion of $\phi(x)/x$, which may not be necessary altogether.

Proof. We will make use of Proposition 3.6.10 and the Kontorovich–Lebedev inversion A.4.3. Let $f(x) = x^{-1}\phi(x)$. Then, f satisfies the conditions for the Kontorovich–Lebedev inversion. However, before we proceed, we shall record a couple of bounds:

$$K_{2it}(x) \ll e^{-\pi|t|}\left(1+|\log(x)|\right), \tag{3.58}$$

$$\int_0^\infty K_{2it}(x)\phi(x)\frac{dx}{x^2} \ll_\phi e^{-\pi|t|}|t|^{-2}(1+|t|)^{-\delta'}, \tag{3.59}$$

$$\int_L K_{it}(\omega q)\left(q+\frac{1}{q}\right)^{2\sigma-2} q^{\pm\kappa-1}dq \ll_Q (1+|\log(\omega)|)(1+|t|)^{-\frac{3}{2}}, \tag{3.60}$$

where $\kappa, t\in\mathbb{R}$ with $|\kappa|<2$, $x,\sigma,\omega,Q\in\mathbb{R}^+$ with $1\le\sigma<1+\frac{1}{4}$, $\omega\le Q$, and some $\delta'>0$. The first estimate (3.58) is rather standard. For the range $|t|\ll 1+x$, we make use of the integral representation (A.16):

$$\begin{aligned}
K_{2it}(x) &= \frac{\Gamma(\frac{1}{2}+2it)(2x)^{2it}}{\sqrt{\pi}}\int_0^\infty \frac{\cos(y)}{(y^2+x^2)^{\frac{1}{2}+2it}}dy \\
&\ll e^{-\pi|t|}\left|\int_0^{x+1}\frac{1}{(y^2+x^2)^{\frac{1}{2}}}dy + \frac{\sin(y)}{(y^2+x^2)^{\frac{1}{2}}}\Big|_{y=x+1}^{\infty} + \left(\tfrac{1}{2}+2it\right)\int_{x+1}^\infty \frac{1}{y^2}dy\right| \\
&\ll e^{-\pi|t|}\left(1+|\log(x)|+\frac{|t|}{1+x}\right).
\end{aligned}$$

The range $|t|\ge x\ge 1$ follows from [BST13, Prop. 2] and the remaining range follows from the Taylor expansion around $x=0$, see [BST13, Section 3.1]. In order to prove the second estimate (3.59), we argue similarly to [DI83, pp. 265,266]. We shall require the integral representation (A.17):

$$K_{2it}(x) = \frac{1}{2\pi i}\int_{(1)} 2^{s-1}x^{-s}\Gamma\left(\tfrac{s}{2}+it\right)\Gamma\left(\tfrac{s}{2}-it\right)ds.$$

We shift the contour to $\mathrm{Re}(s) = -1 - \eta$, where $\eta = \min\{\frac{1}{2}, \frac{\delta}{2}\}$. Along the way, we pick up two simple poles at $s = \pm 2it$. We therefore find

$$K_{2it}(x) = 2^{2it-1}x^{-2it}\Gamma(2it) + 2^{-2it-1}x^{2it}\Gamma(-2it) \\ + \frac{1}{2\pi i}\int_{(-1-\eta)} 2^{s-1}x^{-s}\Gamma\left(\tfrac{s}{2}+it\right)\Gamma\left(\tfrac{s}{2}-it\right)ds.$$

By inserting this, we arrive at

$$\int_0^\infty K_{2it}(x)\phi(x)\frac{dx}{x^2} = 2^{2it-1}\Gamma(2it)\int_0^\infty \phi(x)x^{-2it-2}dx \\ + 2^{-2it-1}\Gamma(-2it)\int_0^\infty \phi(x)x^{2it-2}dx \\ + \int_0^\infty\int_{(-1-\eta)} 2^{s-1}x^{-s-2}\Gamma\left(\tfrac{s}{2}+it\right)\Gamma\left(\tfrac{s}{2}-it\right)ds\phi(x)dx. \quad (3.61)$$

The latter integral converges absolutely and hence we may exchange the order of integration. We recall ϕ has a zero of order at least two at $x = 0$. Hence, integrating by parts shows

$$\int_0^\infty \phi(x)x^{-s-2}dx = \frac{1}{s+1}\int_0^\infty \phi'(x)x^{-s-1}dx = \frac{1}{s(s+1)}\int_0^\infty \phi''(x)x^{-s}dx,$$

where the first equality holds for $-1-\delta < \mathrm{Re}(s) < 1, s \neq -1$, and the second equality for $-\delta < \mathrm{Re}(s) < 1, s \neq -1, 0$. By using this together with $|\Gamma(\pm 2it)| \ll (t\sinh(2\pi t))^{-\frac{1}{2}}$, we find that the first two summands in (3.61) contribute at most $O(e^{-\pi|t|}|t|^{-2}(1+|t|)^{-\frac{1}{2}})$, which is satisfactory. The third summand of (3.61) is equal to

$$\int_{(-1-\eta)} \frac{2^{s-1}\Gamma\left(\frac{s+2}{2}+it\right)\Gamma\left(\frac{s+2}{2}-it\right)}{(s+1)\left(\frac{s}{2}+it\right)\left(\frac{s}{2}-it\right)}\int_0^\infty \phi'(x)x^{-s-1}dxds \\ \ll_\phi \int_{-\infty}^\infty \frac{(\cosh(2\pi t)+\cosh(\pi u))^{-\frac{1}{2}}}{(1+|u|)(1+|2t-u|)^{1+\frac{\eta}{2}}(1+|2t+u|)^{1+\frac{\eta}{2}}}du \\ \ll_\phi e^{-\pi|t|}(1+|t|)^{-2-\frac{\eta}{2}+o(1)},$$

which completes the proof. The third estimate (3.60) is [Pro05, Eqs. (40),(42)].As the latter lacks a proof, we give one here following the argument of the proof of [Pro79, Eq. (54)]. We first replace the K-Bessel function with the I-Bessel function by using (A.9)

$$K_{it}(\omega q) = \frac{\pi i}{2\sinh(\pi t)}\left[I_{it}(\omega q) - I_{-it}(\omega q)\right]$$

and further inserting the Taylor expansion (A.14)

$$I_{\pm it}(\omega q) = \sum_{m=0}^\infty \frac{\left(\frac{\omega q}{2}\right)^{\pm it+2m}}{m!\Gamma(1+m\pm it)}.$$

The whole expression converges absolutely and we may exchange sum and integral. We arrive at

$$\int_L K_{it}(\omega q)\left(q+\frac{1}{q}\right)^{2\sigma-2} q^{\kappa-1}dq$$
$$= \frac{\pi}{2\sinh(\pi t)}\sum_{m=0}^{\infty}\frac{1}{m!}\left[\frac{\left(\frac{\omega}{2}\right)^{2m-it}}{\Gamma(1+m-it)}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{(t+i(2m+\kappa))\theta}\left(2\cos(\theta)\right)^{2\sigma-2}d\theta \right.$$
$$\left. - \frac{\left(\frac{\omega}{2}\right)^{2m+it}}{\Gamma(1+m+it)}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{(-t+i(2m+\kappa))\theta}\left(2\cos(\theta)\right)^{2\sigma-2}d\theta\right].$$

Furthermore, we have for $|t|\geq 1$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{(\pm t+i(2m+k))\theta}\left(2\cos(\theta)\right)^{2\sigma-2}d\theta = \left.\frac{e^{(\pm t+i(2m+k))\theta}}{\pm t+i(2m+k)}\left(2\cos(\theta)\right)^{2\sigma-2}\right|_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}}$$
$$+2(2\sigma-2)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{e^{(\pm t+i(2m+k))\theta}}{\pm t+i(2m+k)}\left(2\cos(\theta)\right)^{2\sigma-3}\sin(\theta)d\theta$$
$$\ll \frac{e^{\frac{\pi}{2}|t|}}{|t|+1}+2(2\sigma-2)\int_0^{\frac{\pi}{2}}\frac{e^{\frac{\pi}{2}|t|}}{|t|+1}\left(2\cos(\theta)\right)^{2\sigma-3}\sin(\theta)d\theta \ll \frac{e^{\frac{\pi}{2}|t|}}{|t|+1}$$

by integration by parts. By further invoking Sterling's approximation for the Gamma function, the claimed bound for $|t|\geq 1$ follows. When $|t|\leq 1$, one needs to group the plus and the minus term as follows

$$\frac{\left(\frac{\omega}{2}\right)^{2m-it}-\left(\frac{\omega}{2}\right)^{2m+it}}{\Gamma(1+m-it)}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{(t+i(2m+\kappa))\theta}\left(2\cos(\theta)\right)^{2\sigma-2}d\theta$$
$$+\left(\frac{\omega}{2}\right)^{2m+it}\left(\frac{1}{\Gamma(1+m-it)}-\frac{1}{\Gamma(1+m+it)}\right)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{(t+i(2m+\kappa))\theta}\left(2\cos(\theta)\right)^{2\sigma-2}d\theta$$
$$\frac{\left(\frac{\omega}{2}\right)^{2m+it}}{\Gamma(1+m+it)}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(e^{(t+i(2m+\kappa))\theta}-e^{(-t+i(2m+\kappa))\theta}\right)\left(2\cos(\theta)\right)^{2\sigma-2}d\theta$$

and make use of the estimates

$$\left(\frac{\omega}{2}\right)^{it}-\left(\frac{\omega}{2}\right)^{-it} \ll t\left|\log\left(\frac{\omega}{2}\right)\right|,$$
$$\frac{1}{\Gamma(1+m-it)}-\frac{1}{\Gamma(1+m+it)} \ll t,$$
$$e^{t\theta}-e^{-t\theta} \ll t\theta,$$

to arrive at the same conclusion.

We now proceed by multiplying the equality in Proposition 3.6.10 by

$$\frac{4}{\pi^2}t\sinh(2\pi t)\int_0^{\infty}K_{2it}(x)\phi(x)\frac{dx}{x^2}$$

and integrate t from $-\infty$ to ∞. The left-hand side is absolutely convergent for $\sigma > 1$ by (3.58), (3.59), and (3.13). Thus we may exchange summation and integral. By applying the Kontorovitch–Lebedev inversion A.4.3, we find that the left-hand side is equal to

$$\frac{1}{4\pi\sqrt{(m+\eta_{\mathfrak{a}})|n+\eta_{\mathfrak{b}}|}} \sum_{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}} \frac{S_{\mathfrak{a},\mathfrak{b}}^{\upsilon,\kappa}(m,n;c)}{c^{2\sigma-1}} \phi\left(\frac{4\pi\sqrt{(m+\eta_{\mathfrak{a}})|n+\eta_{\mathfrak{b}}|}}{c}\right). \tag{3.62}$$

Since $\phi(x) = O(x^2)$ as $x \to 0$, we find that this sum converges absolutely and uniformly for $\sigma \geq 1$. Thus, the limit $\sigma \to 1^+$ exists and is equal to its value at $\sigma = 1$.

We shall argue as in [Pro05] that the right-hand side converges absolutely uniformly in $1 \leq \sigma \leq 1.01$ as well. By applying the inequality between the arithmetic and geometric mean, we find that the right-hand side is bounded by

$$\begin{aligned}
&\frac{(2\pi)^{1-2\sigma}((m+\eta_{\mathfrak{a}})|n+\eta_{\mathfrak{b}}|)^{1-\sigma}}{2\Gamma(2\sigma-1)} \cdot \int_{-\infty}^{\infty} \frac{4}{\pi^2} t\sinh(2\pi t)\left|\int_0^\infty K_{2it}(x)\phi(x)\frac{dx}{x^2}\right| \\
&\qquad \times \Bigg(\sum_{h\in\mathcal{B}_\kappa(\Gamma,\upsilon)} (|\rho_h(\mathfrak{a},m)|^2 + |\rho_h(\mathfrak{b},n)|^2)\Lambda(\sigma+it,\sigma-it,t_h) \\
&\qquad\qquad + \frac{1}{4}\sum_{\mathfrak{c}\text{ sing.}} \int_{-\infty}^{\infty} \frac{|\mathcal{Z}_{\mathfrak{c},\mathfrak{a}}^{\upsilon,\kappa}(0,m;\frac12+ir)|^2}{(m+\eta_{\mathfrak{a}})|\Gamma(\frac12+\frac{\kappa}{2}+ir)|^2}\Lambda(\sigma+it,\sigma-it,r)dr \\
&\qquad\qquad\quad + \frac{1}{4}\sum_{\mathfrak{c}\text{ sing.}} \int_{-\infty}^{\infty} \frac{|\mathcal{Z}_{\mathfrak{c},\mathfrak{b}}^{\upsilon,\kappa}(0,n;\frac12+ir)|^2}{|n+\eta_{\mathfrak{b}}||\Gamma(\frac12-\frac{\kappa}{2}+ir)|^2}\Lambda(\sigma+it,\sigma-it,r)dr\Bigg)dt.
\end{aligned} \tag{3.63}$$

By using Propositions 3.6.8 and 3.6.9, we see that the right-hand side is finite for $1 < \sigma \leq 1.01$ if

$$\int_{-\infty}^{\infty} t\sinh(2\pi t)\left|\int_0^\infty K_{2it}(x)\phi(x)\frac{dx}{x^2}\right|\left|\int_L K_{2it}(\omega q)\left(q+\frac{1}{q}\right)^{2\sigma-2} q^{\pm\kappa-1}dq\right| |\Gamma(\epsilon+it)|^2 dt$$

is $\ll_{\epsilon,Q} \omega^{o(1)}$ for $0 < \omega \leq Q$ and $0 < \epsilon < \frac12$ and

$$\int_{-\infty}^{\infty} t\sinh(2\pi t)\left|\int_0^\infty K_{2it}(x)\phi(x)\frac{dx}{x^2}\right| |\Gamma(\epsilon+it)|^2 dt < \infty.$$

Both are true by virtue of (3.59), (3.60), and Stirling approximation.

In the next step, we show that this convergence carries on to $1 \leq \sigma \leq 1.01$. We separate the Maass forms for which $t_h \notin \mathbb{R}$. There are only finitely many of them and for each of them the limit $\sigma \to 1^+$ exists. The only problem arises when $\kappa = 0$ and $t_h = \pm\frac12 i$, in which case h is constant and thus h has Fourier coefficient $\rho_h(\mathfrak{a},m) = 0$ for non-zero m. For $t_h, r \in \mathbb{R}$, we have by Stirling approximation that

$$\Lambda(\sigma_1+it,\sigma_1-it;r) \ll \Lambda(\sigma_2+it,\sigma_2-it;r), \quad \forall 1 \leq \sigma_1 \leq \sigma_2 \leq 1.01, \forall t,r \in \mathbb{R}.$$

Hence, we have the proclaimed convergence. We further require another integral which is recorded in [Pro79, Eq. (39)]

$$\frac{4}{\pi^2}\int_{-\infty}^{\infty}\Lambda(1+it,1-it,r)t\sinh(2\pi t)\int_0^\infty K_{2it}(x)\phi(x)\frac{dx}{x^2}dt = 4\int_0^\infty K_{2ir}(x)\phi(x)\frac{dx}{x}$$
$$= \frac{\pi}{\cosh(\pi r)}\check{\phi}(r).$$

By pulling the integral over t in the inside, we find that the limit as $\sigma \to 1^+$ of the right-hand side is equal to

$$\frac{1}{2}\sum_{h\in\mathcal{B}_\kappa(\Gamma,\upsilon)}\frac{\overline{\rho_h(\mathfrak{a},m)}\rho_h(\mathfrak{b},n)}{\cosh(\pi t_h)}\check{\phi}(t_h) + \frac{1}{8\sqrt{(m+\eta_{\mathfrak{a}})|n+\eta_{\mathfrak{b}}|}}$$
$$\times\sum_{\mathfrak{c}\text{ sing.}}\int_{-\infty}^{\infty}\left|\frac{m+\eta_{\mathfrak{a}}}{n+\eta_{\mathfrak{b}}}\right|^{-ir}\frac{\overline{\mathcal{Z}^{\upsilon,\kappa}_{\mathfrak{c},\mathfrak{a}}(0,m;\frac{1}{2}+ir)}}{\Gamma(\frac{1}{2}+\frac{\kappa}{2}-ir)}\frac{\mathcal{Z}^{\upsilon,\kappa}_{\mathfrak{c},\mathfrak{b}}(0,n;\frac{1}{2}+ir)}{\Gamma(\frac{1}{2}-\frac{\kappa}{2}+ir)}\frac{\check{\phi}(r)dr}{\cosh(\pi r)}.$$

Equating this with (3.62) yields the theorem. □

4

THE TWISTED LINNIK CONJECTURE

Cancellation in the sum

$$\sum_{\substack{c \in \mathcal{C}_{\mathfrak{a},\mathfrak{b}} \\ c \le C}} \frac{S^{\upsilon,\kappa}_{\mathfrak{a},\mathfrak{b}}(m,n;c)}{c}, \tag{4.1}$$

is inherently linked to analytic properties of the Kloosterman zeta function $\mathcal{Z}^{\upsilon,\kappa}_{\mathfrak{a},\mathfrak{b}}(m,n;s)$. It can be shown that if there is a pole s_0 in the half-plane $\mathrm{Re}(s) > \frac{1}{2}$, then s_0 is real and there is a Maass form with eigenvalue $s_0(1-s_0)$. Conversely, one can show that if there is a Maass form with eigenvalue $s_0(1-s_0)$, then there is an $n \in \mathbb{Z}$ such that $\mathcal{Z}^{\upsilon,\kappa}_{\infty,\infty}(n,n;s)$ has a pole at $s = s_0$. Consequently, if one can show that (4.1) is $O(C^{2t})$, then the smallest eigenvalue is at least $\frac{1}{4} - t^2$. This is how Selberg [Sel65] showed that for congruence subgroups the smallest eigenvalue is at least $\frac{3}{16}$, by invoking the Weil bound for the (classical) Kloosterman sums. If one were to consider a smooth cut-off in (4.1) instead of a sharp cut-off the above discussion becomes an equivalence.

Taking into account that the (classical) Kloosterman sums are thought to undergo random sign changes, one ought to believe that the sum (4.1) is $C^{o(1)}$ and hence the smallest eigenvalue is at least $\frac{1}{4}$ for congruence subgroups. The latter is know as the Selberg eigenvalue conjecture and the former as the Linnik–Selberg conjecture on sums of Kloosterman sums. We shall state the latter conjecture with the necessary ϵ safety factor, which was pointed out by Sarnak–Tsimerman [ST09].

Conjecture 4.0.1. *Let $m, n \in \mathbb{N}$. Then, we have*

$$\sum_{c \le C} \frac{S(m,n;c)}{c} \ll_\epsilon (mnC)^\epsilon.$$

The range $C \ge \sqrt{mn}$ is known as the Linnik range and the range $C \le \sqrt{mn}$ is known as the Selberg range. The latter poses much more difficulty than the former as we shall see. The first and basically the only improvement over what the Weil bound tells you was achieved by Kuznetsov [Kuz80] by establishing and exploiting his eponymous trace formula. He showed

$$\sum_{c\leq C}\frac{S(m,n;c)}{c}\ll_{m,n}C^{\frac{1}{6}}\log(2C)^{\frac{1}{3}}.$$

The dependence on m,n, although not stated, can be made explicit in a trivial manner by adding the term $(mn)^{\frac{1}{4}+\epsilon}$. This comes from estimating the Selberg range trivially. Sarnak–Tsimerman [ST09] improved upon this. They showed

$$\sum_{c\leq C}\frac{S(m,n;c)}{c}\ll_{\epsilon}(mnC)^{\epsilon}\left(C^{\frac{1}{6}}+(mn)^{\frac{1}{3}}+(m+n)^{\frac{1}{8}}(mn)^{\frac{\theta}{2}}\right).$$

This result has been further generalised by Ganguly–Sengupta [GS12] to arithmetic progressions $c\equiv 0\bmod(N)$ and by Blomer–Milićević [BM15a] to progressions $c\equiv a\bmod(N)$ with $(a,N)=1$. Motivated by an application to intrinsic Diophantine approximation, which we shall learn more about in Section 6.3, Browning–Kumaraswamy–Steiner [BKS17] have proposed the following extension of the Linnik–Selberg conjecture.

Conjecture 4.0.2 (Twisted Linnik–Selberg)**.** *Let $B\geq 1$ and let $m,n\in\mathbb{Z}$ be non-zero. Let $N\in\mathbb{N}$ and let $a\in\mathbb{Z}/N\mathbb{Z}$. Then, for any $\alpha\in[-B,B]$ we have*

$$\sum_{\substack{c\leq C\\ c\equiv a\bmod N}}\frac{S(m,n;c)}{c}e\left(\frac{2\sqrt{mn}}{c}\alpha\right)\ll_{\epsilon,N,B}(|mn|C)^{\epsilon},$$

for any $\epsilon>0$.

In this chapter, we shall recollect the progress that has been made towards this conjecture by the author in [Ste17]. Let us introduce some simplifying notation: $F\lesssim G$ means $|F|\leq K_{\epsilon}(CmnN(1+|\alpha|))^{\epsilon}G$ for some positive constant K_{ϵ}, depending on ϵ, and every $\epsilon>0$. The main theorem of [Ste17] reads as follows.

Theorem 4.0.1. *Let $C\geq 1$, $\alpha\in\mathbb{R}$, $N\in\mathbb{N}$, and $m,n\in\mathbb{Z}$ with $mn>0$, $(m,n,N)=1$, and $N\ll\min\{(mn)^{\frac{1}{4}},C^{\frac{1}{2}}\}$. Then, we have*

$$\begin{aligned}&\sum_{\substack{c\leq C\\ c\equiv 0\bmod(N)}}\frac{S(m,n;c)}{c}e\left(\frac{2\sqrt{mn}}{c}\alpha\right)\\ &\qquad+2\pi\sum_{\substack{h\in\mathcal{B}_0(\Gamma_0(N),1)\\ t_h\in i[0,\theta]}}\frac{\sqrt{mn}\cdot\overline{\rho_h(\infty,m)}\rho_h(\infty,n)}{\cos(\pi|t_h|)}\int_{\frac{\sqrt{mn}}{C}}^{\infty}Y_{2|t_h|}(x)e^{i\alpha x}\frac{dx}{x}\\ &\qquad\lesssim\frac{C^{\frac{1}{6}}}{N^{\frac{1}{3}}}+(1+|\alpha|^{\frac{1}{3}})\frac{(mn)^{\frac{1}{6}}}{N^{\frac{2}{3}}}+\frac{m^{\frac{1}{4}}(m,N)^{\frac{1}{4}}+n^{\frac{1}{4}}(n,N)^{\frac{1}{4}}}{N^{\frac{1}{2}}}\\ &\qquad\qquad+\min\left\{\frac{(mn)^{\frac{1}{8}+\frac{\theta}{2}}(mn,N)^{\frac{1}{8}}}{N^{\frac{1}{2}}},\frac{(mn)^{\frac{1}{4}}(mn,N)^{\frac{1}{4}}}{N}\right\},\end{aligned}$$

where Y_t is the Bessel function of the second kind of order t and θ is the best known progress towards the Ramanujan–Selberg conjecture.

A few remarks about this theorem are in order. First, we should remark that one has $\theta \leq \frac{7}{64}$ by the work of Kim–Sarnak [Kim03]. Next, we observe the appearance of a main term, which is contrary to [GS12]. Indeed, the latter has an erroneous treatment of the exceptional spectrum[a]. One may further analyse the main term by making use of asymptotics of the Bessel function of the second kind $Y_t(y)$ for $y \to 0$. However, the reader familiar with Bessel functions may know that these asymptotics behave quite differently for $t = 0$ and $t > 0$ and therefore it would generate uniformity issues in the parameter N. One may also bound the main term altogether. In this case, one gets the following corollary.

Corollary 4.0.2. *Assume the same assumptions as in Theorem 4.0.1. Then*

$$\sum_{\substack{c \leq C \\ c \equiv 0 \bmod (N)}} \frac{S(m,n;c)}{c} e\left(\frac{2\sqrt{mn}}{c}\alpha\right)$$
$$\lesssim \frac{C^{\frac{1}{6}}}{N^{\frac{1}{3}}} + C^{2\theta} + (1+|\alpha|^{\frac{1}{3}})\frac{(mn)^{\frac{1}{6}}}{N^{\frac{2}{3}}} + \frac{m^{\frac{1}{4}}(m,N)^{\frac{1}{4}} + n^{\frac{1}{4}}(n,N)^{\frac{1}{4}}}{N^{\frac{1}{2}}}$$
$$+ \min\left\{\frac{(mn)^{\frac{1}{8}+\frac{\theta}{2}}(mn,N)^{\frac{1}{8}}}{N^{\frac{1}{2}}}, \frac{(mn)^{\frac{1}{4}}(mn,N)^{\frac{1}{4}}}{N}\right\}.$$

As far as the restrictions go in Theorem 4.0.1, they are not very limiting. Indeed, if $N \geq C^{\frac{1}{2}}$, then the Weil bound, which gives the bound $N^{-1}C^{\frac{1}{2}+\epsilon}$, is more than sufficient, and if $(mn)^{\frac{1}{4}} \leq N \leq C^{\frac{1}{2}}$, then one is automatically in the easier Linnik range and for instance the holomorphic contribution is negligible. One may also consider $mn < 0$, which would lead one to analyse different Bessel transforms; or incorporate the further restriction $c \equiv a \bmod (N')$ with $(a, N') = 1$ as in [BM15a]. However, for the latter, if one follows the simplification suggested by Kiral–Young [KY17], then an analogue to Proposition 4.3.1 for the group $\Gamma_0(N) \cap \Gamma_1(N')$ is required. In fact, the associated Kloosterman sums for this group admit further cancellation as can be seen in [Hum16], thereby leading to stronger results in terms of the parameter N'.

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a The compact domain to which they apply the mean value theorem of calculus varies and this may not be circumvented, since if the exceptional spectrum is non-empty, then the function they consider has a pole at 0.

For $|\alpha| < 1$, one may improve Theorem 4.0.1 slightly, thereby recovering the results of [ST09] and [GS12].

Theorem 4.0.3. *Let $C \geq 1$, $\alpha \in \mathbb{R}$ with $|\alpha| < 1$, $N \in \mathbb{N}$ and $m, n \in \mathbb{Z}$ with $mn > 0$, $N \ll \min\{(mn)^{\frac{1}{4}}, C^{\frac{1}{2}}\}$, and $(m,n,N) = 1$. Then, we have*

$$\sum_{\substack{c \leq C \\ c \equiv 0 \bmod (N)}} \frac{S(m,n;c)}{c} e\left(\frac{2\sqrt{mn}}{c}\alpha\right) + 2\pi \sum_{\substack{h \in \mathcal{B}_0(\Gamma_0(N),1) \\ t_h \in i[0,\theta]}} \frac{\sqrt{mn} \cdot \overline{\rho_h(\infty,m)}\rho_h(\infty,n)}{\cos(\pi|t_h|)} \int_{\frac{\sqrt{mn}}{C}}^{\infty} Y_{2|t_h|}(x) e^{i\alpha x} \frac{dx}{x}$$
$$\lesssim (1-|\alpha|)^{-\frac{1}{2}-\epsilon} \left(\frac{m^{\frac{1}{8}}(m,N)^{\frac{1}{8}} + n^{\frac{1}{8}}(n,N)^{\frac{1}{8}}}{N^{\frac{1}{4}}} \min\left\{(mn)^{\frac{\theta}{2}}, \frac{m^{\frac{1}{8}}(m,N)^{\frac{1}{8}} + n^{\frac{1}{8}}(n,N)^{\frac{1}{8}}}{N^{\frac{1}{4}}}\right\} + \frac{C^{\frac{1}{6}}}{N^{\frac{1}{3}}} + \frac{(mn)^{\frac{1}{6}}}{N^{\frac{2}{3}}} + \min\left\{\frac{(mn)^{\frac{1}{16}+\frac{3\theta}{4}}(mn,N)^{\frac{1}{16}}}{N^{\frac{1}{4}}}, \frac{(mn)^{\frac{1}{4}}(mn,N)^{\frac{1}{4}}}{N}\right\}\right)$$

and

$$\sum_{\substack{c \leq C \\ c \equiv 0 \bmod (N)}} \frac{S(m,n;c)}{c} e\left(\frac{2\sqrt{mn}}{c}\alpha\right)$$
$$\lesssim (1-|\alpha|)^{-\frac{1}{2}-\epsilon} \left(\frac{m^{\frac{1}{8}}(m,N)^{\frac{1}{8}} + n^{\frac{1}{8}}(n,N)^{\frac{1}{8}}}{N^{\frac{1}{4}}} \min\left\{(mn)^{\frac{\theta}{2}}, \frac{m^{\frac{1}{8}}(m,N)^{\frac{1}{8}} + n^{\frac{1}{8}}(n,N)^{\frac{1}{8}}}{N^{\frac{1}{4}}}\right\} + \frac{C^{\frac{1}{6}}}{N^{\frac{1}{3}}} + \frac{(mn)^{\frac{1}{6}}}{N^{\frac{2}{3}}} + \min\left\{\frac{(mn)^{\frac{1}{16}+\frac{3\theta}{4}}(mn,N)^{\frac{1}{16}}}{N^{\frac{1}{4}}}, \frac{(mn)^{\frac{1}{4}}(mn,N)^{\frac{1}{4}}}{N}\right\}\right) + C^{2\theta}.$$

Rather than proving 4.0.1 directly, we shall prove a dyadic version thereof from which we may deduce Theorem 4.0.1.

Theorem 4.0.4. *Let $\alpha \in \mathbb{R}$, $N \in \mathbb{N}$ and $m, n \in \mathbb{Z}$ with $mn > 0$ and $(m,n,N) = 1$. Assume $N \ll \min\{(mn)^{\frac{1}{4}}, C^{\frac{1}{2}}\}$. Then, we have*

$$\sum_{\substack{C \leq c < 2C \\ c \equiv 0 \bmod (N)}} \frac{S(m,n;c)}{c} e\left(\frac{2\sqrt{mn}}{c}\alpha\right) + 2\pi \sum_{\substack{h \in \mathcal{B}_0(\Gamma_0(N),1) \\ t_h \in i[0,\theta]}} \frac{\sqrt{mn} \cdot \overline{\rho_h(\infty,m)}\rho_h(\infty,n)}{\cos(\pi|t_h|)} \int_{\frac{X}{2}}^{X} Y_{2|t_h|}(x) e^{i\alpha x} \frac{dx}{x}$$
$$\lesssim \frac{C^{\frac{1}{6}}}{N^{\frac{1}{3}}} + (1+|\alpha|)\frac{(mn)^{\frac{1}{2}}}{C} + \frac{m^{\frac{1}{4}}(m,N)^{\frac{1}{4}} + n^{\frac{1}{4}}(n,N)^{\frac{1}{4}}}{N^{\frac{1}{2}}} + \min\left\{\frac{(mn)^{\frac{\theta}{2}+\frac{1}{8}}(mn,N)^{\frac{1}{8}}}{N^{\frac{1}{2}}}, \frac{(mn)^{\frac{1}{4}}(mn,N)^{\frac{1}{4}}}{N}\right\}.$$

For $|\alpha| < 1$, we can do slightly better:

$$\begin{aligned}
&\sum_{\substack{C \le c < 2C \\ c \equiv 0 \bmod (N)}} \frac{S(m,n;c)}{c} e\left(\frac{2\sqrt{mn}}{c}\alpha\right) \\
&\qquad + 2\pi \sum_{\substack{h \in \mathcal{B}_0(\Gamma_0(N),1) \\ t_h \in i[0,\theta]}} \frac{\sqrt{mn} \cdot \overline{\rho_h(\infty,m)}\rho_h(\infty,n)}{\cos(\pi|t_h|)} \int_{\frac{X}{2}}^{X} Y_{2|t_h|}(x) e^{i\alpha x} \frac{dx}{x} \\
&\lesssim (1-|\alpha|)^{-\frac{1}{2}-\epsilon} \left(\frac{m^{\frac{1}{8}}(m,N)^{\frac{1}{8}} + n^{\frac{1}{8}}(n,N)^{\frac{1}{8}}}{N^{\frac{1}{4}}} \min\left\{ (mn)^{\frac{\theta}{2}}, \frac{m^{\frac{1}{8}}(m,N)^{\frac{1}{8}} + n^{\frac{1}{8}}(n,N)^{\frac{1}{8}}}{N^{\frac{1}{4}}} \right\} \right. \\
&\qquad \left. + \frac{C^{\frac{1}{6}}}{N^{\frac{1}{3}}} + \frac{(mn)^{\frac{1}{2}}}{C} + \min\left\{ \frac{(mn)^{\frac{3\theta}{4}+\frac{1}{16}}(mn,N)^{\frac{1}{16}}}{N^{\frac{1}{4}}}, \frac{(mn)^{\frac{1}{4}}(mn,N)^{\frac{1}{4}}}{N} \right\} \right).
\end{aligned}$$

The proof is taken straight from the author's work [Ste17] and follows the argument of [ST09] and [GS12]. We replace the sharp cut-off with a smooth cut-off and use the Kuznetsov trace formula. We shall restate the Kuznetsov trace formula 3.10.1 for the Fuchsian group $\Gamma_0(N)$ with trivial multiplier system of weight 0.

Proposition 4.0.5 (Kuznetsov trace formula). *Let $N \in \mathbb{N}$ and $m, n \in \mathbb{Z}$ be two integers with $mn > 0$. Then, for any C^3-class function f with compact support in $]0,\infty[$ one has*

$$\sum_{c \equiv 0 \bmod (N)} \frac{S(m,n;c)}{c} f\left(\frac{4\pi\sqrt{mn}}{c}\right) = \mathcal{H}_N(m,n;f) + \mathcal{M}_N(m,n;f) + \mathcal{E}_N(m,n;f),$$

where

$$\begin{aligned}
\mathcal{H}_N(m,n;f) &= \frac{1}{\pi} \sum_{\substack{k \equiv 0 \bmod (2) \\ k>0}} \sum_{h \in \mathcal{B}_k(\Gamma_0(N),1)} \frac{(-1)^{\frac{k}{2}}\Gamma(k)}{(4\pi\sqrt{mn})^{k-1}} \overline{\psi_h(\infty,|m|)}\psi_h(\infty,|n|)\widetilde{f}(k-1), \\
\mathcal{M}_N(m,n;f) &= 4\pi \sum_{h \in \mathcal{B}_0(\Gamma_0(N),1)} \frac{\sqrt{mn}}{\cosh(\pi t_h)} \overline{\rho_h(\infty,|m|)}\rho_h(\infty,|n|)\widehat{f}(t_h), \\
\mathcal{E}_N(m,n;f) &= \sum_{\mathfrak{c} \text{ sing.}} \int_{-\infty}^{\infty} \left(\frac{n}{m}\right)^{it} \overline{\mathcal{Z}^{1,0}_{\mathfrak{c},\infty}(0,|m|;\tfrac{1}{2}+it)}\mathcal{Z}^{1,0}_{\mathfrak{c},\infty}(0,|n|;\tfrac{1}{2}+it)\widehat{f}(t)dt,
\end{aligned}$$

and the transforms are given by

$$\begin{aligned}
\widetilde{f}(t) &= \int_0^\infty J_t(y) f(y) \frac{dy}{y}, \\
\widehat{f}(t) &= \frac{i}{\sinh(\pi t)} \int_0^\infty \frac{J_{2it}(x) - J_{-2it}(x)}{2} f(x) \frac{dx}{x}.
\end{aligned}$$

We set $f(x) = e^{i\alpha x}g(x)$ with g smooth real-valued bump function satisfying the following properties

$$\begin{aligned} &g(x) = 1 \text{ for } \frac{2\pi\sqrt{mn}}{C} \le x \le \frac{4\pi\sqrt{mn}}{C}, \\ &g(x) = 0 \text{ for } x \le \frac{2\pi\sqrt{mn}}{C+T} \text{ and } x \ge \frac{4\pi\sqrt{mn}}{C-T}, \\ &\|g'\|_1 \ll 1 \text{ and } \|g''\|_1 \ll \frac{C}{X\cdot T}, \end{aligned} \tag{4.2}$$

where

$$X = \frac{4\pi\sqrt{mn}}{C} \text{ and } 1 \le T \le \frac{C}{2}$$

is a parameter to be chosen at a later point. Note that we have $\operatorname{Supp} g \subseteq [\frac{X}{3}, 2X]$. We now wish to compare the smooth sum

$$\sum_{c\equiv 0 \bmod (N)} \frac{S(m,n;c)}{c} f\left(\frac{4\pi\sqrt{mn}}{c}\alpha\right) \tag{4.3}$$

with the sharp cut-off in Theorem 4.0.4. By making use of the Weil bound for the Kloosterman sum, we find that their difference is bounded by

$$\begin{aligned} \sum_{\substack{C-T\le c\le C \text{ or} \\ 2C\le c\le 2C+2T, \\ c\equiv 0 \bmod (N)}} \frac{1}{c}|S(m,n;c)| &\le \sum_{\substack{C-T\le c\le C \text{ or} \\ 2C\le c\le 2C+2T \\ c\equiv 0 \bmod (N)}} \frac{\tau(c)}{\sqrt{c}}(m,n,c)^{\frac{1}{2}} \\ &\le \frac{\tau(N)}{\sqrt{N}} \sum_{e|(m,n)} \sum_{\substack{\frac{C-T}{Ne}\le c'\le \frac{C}{Ne} \text{ or} \\ \frac{2C}{Ne}\le c'\le \frac{2C+2T}{Ne}}} \frac{\tau(ec')}{\sqrt{ec'}} e^{\frac{1}{2}} \\ &\lesssim \frac{1}{\sqrt{N}} \sum_{e|(m,n)} \frac{\sqrt{Ne}}{\sqrt{C}}\left(1+\frac{T}{Ne}\right) \\ &\lesssim \frac{1}{\sqrt{C}}\left((m,n)^{\frac{1}{2}} + \frac{T}{N}\right). \end{aligned} \tag{4.4}$$

Now, we apply the Kuznetsov trace formula (see Proposition 4.0.5) to the smooth sum (4.3). This leads to the expression

$$\sum_{c\equiv 0 \bmod (N)} \frac{S(m,n;c)}{c} f\left(\frac{4\pi\sqrt{mn}}{c}\right) = \mathcal{H}_N(m,n;f) + \mathcal{M}_N(m,n;f) + \mathcal{E}_N(m,n;f).$$

We shall deal with each of these terms separately. In what follows, we shall use many estimates on the Bessel transforms of f, which we shall summarise here, but postpone their proof until Section 4.5.

Lemma 4.0.6. *Let f be as in (4.2). Then, we have*

$$\widehat{f}(t), \widetilde{f}(t) \ll \frac{1+|\log(X)|+\log^{+}(|\alpha|)}{1+X^{\frac{1}{2}}+||\alpha|^2-1|^{\frac{1}{2}}X}, \qquad \forall t \in \mathbb{R}, \tag{4.5}$$

$$\widehat{f}(it) = -\frac{1}{2}\int_{\frac{X}{2}}^{X} Y_{2t}(x)e^{i\alpha x}\frac{dx}{x} + O_{\epsilon,\delta}\left(1+\frac{T}{C}X^{-2t-\epsilon}\right), \qquad \forall 0 \leq t \leq \frac{1}{4}-\delta, \tag{4.6}$$

where $\log^{+}(x) = \max\{0, \log(x)\}$. *For* $t \geq 8$, *we have*

$$\int_0^{\frac{t}{2}} J_t(y)f(y)\frac{dy}{y} \ll \mathbb{1}_{[2X/3,\infty[}(t)\cdot t^{-\frac{1}{2}}e^{-\frac{2}{5}t}, \tag{4.7}$$

$$\int_{\frac{t}{2}}^{t-t^{\frac{1}{3}}} J_t(y)f(y)\frac{dy}{y} \ll \mathbb{1}_{[\frac{1}{4},\infty[}(X)\mathbb{1}_{[X/3,4X]}(t)\cdot t^{-1}(\log(t))^{\frac{2}{3}}, \tag{4.8}$$

$$\int_{t-t^{\frac{1}{3}}}^{t+t^{\frac{1}{3}}} J_t(y)f(y)\frac{dy}{y} \ll \mathbb{1}_{[\frac{1}{4},\infty[}(X)\mathbb{1}_{[3X/16,3X]}(t)\cdot t^{-1}, \tag{4.9}$$

$$\int_{t+t^{\frac{1}{3}}}^{\infty} J_t(y)f(y)\frac{dy}{y} \ll \mathbb{1}_{[\frac{1}{4},\infty[}(X)\mathbb{1}_{[0,3X/2]}(t)\cdot t^{-1}\min\left\{1+|1-|\alpha||^{-\frac{1}{4}}, \left(\frac{X}{t}\right)^{\frac{1}{2}}\right\}, \tag{4.10}$$

where $\mathbb{1}_{\mathcal{I}}$ is the characteristic function of the interval $\mathcal{I}$. Finally, when $|t| \geq 1$ and either $|t| \notin \left[\frac{1}{12}||\alpha|^2-1|^{\frac{1}{2}}X, 2||\alpha|^2-1|^{\frac{1}{2}}X\right]$ or $|\alpha| \leq 1$ we have

$$\widehat{f}(t) \ll |t|^{-\frac{3}{2}}\left(1+\min\left\{\left(\frac{X}{|t|}\right)^{\frac{1}{2}}, ||\alpha|^2-1|^{-1}\left(\frac{X}{|t|}\right)^{-\frac{3}{2}}\right\}\right), \tag{4.11}$$

$$\widehat{f}(t) \ll \frac{C}{T}|t|^{-\frac{5}{2}}\left(1+\min\left\{\left(\frac{X}{|t|}\right)^{\frac{3}{2}}, ||\alpha|^2-1|^{-2}\left(\frac{X}{|t|}\right)^{-\frac{5}{2}}\right\}\right). \tag{4.12}$$

One should mention that similar estimates have been derived previously by Jutila [Jut99] for a slightly different class of functions and ranges.

4.1 THE CONTINUOUS SPECTRUM

The goal of this section is to prove the following bound on the continuous contribution

$$\mathcal{E}_N(m,n;f) \lesssim 1. \tag{4.13}$$

For this endeavour, we require the following lemma.

Lemma 4.1.1. *Let $N = N_\star N_\square^2$ with $N_\star$ square-free and let m, n positive integers. We have*

$$\sum_{\mathfrak{c}\ sing.} \left(\frac{n}{m}\right)^{it} \overline{\mathcal{Z}^{1,0}_{\mathfrak{c},\infty}(0,m;\tfrac{1}{2}+it)}\mathcal{Z}^{1,0}_{\mathfrak{c},\infty}(0,n;\tfrac{1}{2}+it)$$

$$\ll_\epsilon \frac{(m,N_\star N_\square)^{\frac{1}{2}}(n,N_\star N_\square)^{\frac{1}{2}}}{N_\star N_\square}(mnN(1+|t|))^\epsilon.$$

Proof. This is part of [BM15a, Lemma 1]. □

Substituting this inequality into (4.13) yields the bound

$$\begin{aligned}\mathcal{E}_N(m,n;f) &\lesssim \frac{(m,N_\star N_\square)^{\frac{1}{2}}(n,N_\star N_\square)^{\frac{1}{2}}}{N_\star N_\square}\int_{-\infty}^{\infty}(1+|t|)^{\epsilon}|\widehat{f}(t)|dt \\ &\lesssim \int_{-\infty}^{\infty}(1+|t|)^{\epsilon}|\widehat{f}(t)|dt.\end{aligned}$$

We split the integral up into three parts:

$$\begin{aligned}\mathcal{I}_1 &= \pm[\tfrac{1}{12}||\alpha|^2-1|^{\frac{1}{2}}X, 2||\alpha|^2-1|^{\frac{1}{2}}X], \\ \mathcal{I}_2 &= [-\max\{1,X^{\frac{1}{2}}\},\max\{1,X^{\frac{1}{2}}\}]\backslash\mathcal{I}_1, \\ \mathcal{I}_3 &= \pm[\max\{1,X^{\frac{1}{2}}\},\infty[\backslash\mathcal{I}_1.\end{aligned}$$

For $\mathcal{I}_1$, we use (4.5) and arrive at

$$\begin{aligned}\int_{\mathcal{I}_1}(1+|t|)^{\epsilon}|\widehat{f}(t)|dt &\ll_\epsilon \int_{\mathcal{I}_1}(1+|t|)^{\epsilon}\frac{1+|\log(X)|+\log^+(|\alpha|)}{||\alpha|^2-1|^{\frac{1}{2}}X}dt \\ &\ll_\epsilon (1+X)^{\epsilon}(1+|\alpha|)^{\epsilon}(1+|\log(X)|+\log^+(|\alpha|)) \\ &\lesssim 1.\end{aligned}$$

For $\mathcal{I}_2$, we use (4.5) again and arrive at

$$\begin{aligned}\int_{\mathcal{I}_2}(1+|t|)^{\epsilon}|\widehat{f}(t)|dt &\ll_\epsilon \int_{\mathcal{I}_2}(1+|t|)^{\epsilon}\frac{1+|\log(X)|+\log^+(|\alpha|)}{1+X^{\frac{1}{2}}}dt \\ &\ll_\epsilon (1+X)^{\epsilon}(1+|\log(X)|+\log^+(|\alpha|)) \\ &\lesssim 1.\end{aligned}$$

For $\mathcal{I}_3$, we use (4.11) and arrive at

$$\begin{aligned}\int_{\mathcal{I}_3}(1+|t|)^{\epsilon}|\widehat{f}(t)|dt &\ll_\epsilon \int_{\mathcal{I}_3}|t|^{-\frac{3}{2}+\epsilon}\left(1+\left(\frac{X}{|t|}\right)^{\frac{1}{2}}\right)dt \\ &\ll_\epsilon \min\{1,X^{-\frac{1}{4}+\epsilon}\}+X^{\frac{1}{2}}\min\{1,X^{-\frac{1}{2}+\epsilon}\} \\ &\lesssim 1.\end{aligned}$$

This concludes the proof of (4.13).

4.2 THE HOLOMORPHIC SPECTRUM

The goal of this section is to prove the following inequality:

$$\mathcal{H}_N(m,n;f) \lesssim 1+X. \tag{4.14}$$

In order to prove this inequality, we choose our orthonormal basis as in (3.52). Then, $\mathcal{H}_N(m,n;f)$ is equal to and further bounded by

$$\frac{1}{\pi}\sum_{\substack{k\equiv 0 \bmod (2)\\ k>0}}\sum_{M|N}\sum_{\substack{h \text{ new}\\ \text{of level } M}}\sum_{d|\frac{N}{M}}\frac{i^k\Gamma(k)}{(4\pi\sqrt{mn})^{k-1}}\overline{\psi_{h^d}(\infty,m)}\psi_{h^d}(\infty,n)\widetilde{f}(k-1)$$

$$\lesssim \sum_{\substack{k\equiv 0 \bmod (2)\\ k>0}}\sum_{M|N}\sum_{\substack{h \text{ new}\\ \text{of level } M}}\sum_{d|\frac{N}{M}}\frac{\Gamma(k)}{(4\pi)^{k-1}}\frac{N}{M}|\psi_h(\infty,1)|^2|\widetilde{f}(k-1)|$$

$$\lesssim \sum_{\substack{k\equiv 0 \bmod (2)\\ k>0}}\sum_{M|N}\sum_{\substack{h \text{ new}\\ \text{of level } M}}\frac{1}{M}|\widetilde{f}(k-1)|$$

$$\lesssim \sum_{\substack{k\equiv 0 \bmod (2)\\ k>0}} k^{1+\epsilon}|\widetilde{f}(k-1)|,$$

where we have made use of (3.55), (3.56), and $\dim \mathcal{S}_k(\Gamma_0(M),1) \lesssim Mk$ (see Proposition 3.7.1). We split up the latter sum into $k\le 9$ and $k>9$. By using (4.5), we find

$$\sum_{\substack{k\equiv 0 \bmod (2)\\ 9\ge k>0}} k^{1+\epsilon}|\widetilde{f}(k-1)| \ll 1+|\log(X)|+\log^+(|\alpha|)\lesssim 1.$$

We also find

$$\sum_{\substack{k\equiv 0 \bmod (2)\\ k>9}} k^{1+\epsilon}|\widetilde{f}(k-1)| \le \mathcal{S}_1+\mathcal{S}_2+\mathcal{S}_3+\mathcal{S}_4,$$

where

$$\mathcal{S}_1 = \sum_{\substack{k\equiv 0 \bmod (2)\\ k>9}} k^{1+\epsilon}\left|\int_0^{\frac{k-1}{2}} J_{k-1}(y)f(y)\frac{dy}{y}\right|,$$

$$\mathcal{S}_2 = \sum_{\substack{k\equiv 0 \bmod (2)\\ k>9}} k^{1+\epsilon}\left|\int_{\frac{k-1}{2}}^{(k-1)-(k-1)^{\frac{1}{3}}} J_{k-1}(y)f(y)\frac{dy}{y}\right|,$$

$$\mathcal{S}_3 = \sum_{\substack{k\equiv 0 \bmod (2)\\ k>9}} k^{1+\epsilon}\left|\int_{(k-1)-(k-1)^{\frac{1}{3}}}^{(k-1)+(k-1)^{\frac{1}{3}}} J_{k-1}(y)f(y)\frac{dy}{y}\right|,$$

$$\mathcal{S}_4 = \sum_{\substack{k\equiv 0 \bmod (2)\\ k>9}} k^{1+\epsilon}\left|\int_{(k-1)+(k-1)^{\frac{1}{3}}}^{\infty} J_{k-1}(y)f(y)\frac{dy}{y}\right|.$$

By using (4.7), we find

$$\mathcal{S}_1 \ll_\epsilon \sum_{k>9} k^{\frac{1}{2}+\epsilon}e^{-\frac{2}{5}k} \ll_\epsilon 1.$$

By using (4.8), we find

$$\mathcal{S}_2 \ll_\epsilon \sum_{X/3 \le k-1 \le 4X} k^\epsilon \lesssim 1 + X.$$

By using (4.9), we find

$$\mathcal{S}_3 \ll_\epsilon \sum_{3X/16 \le k-1 \le 3X} k^\epsilon \lesssim 1 + X.$$

By using (4.10), we find

$$\mathcal{S}_4 \ll_\epsilon \sum_{3X/2 \ge k-1 > 8} k^\epsilon \left(\frac{X}{k}\right)^{\frac{1}{2}} \lesssim 1 + X.$$

The claimed inequality (4.14) now follows.

4.3 THE NON-HOLOMORPHIC SPECTRUM

In this section, we shall prove the following two estimates

$$\begin{aligned} &\mathcal{M}_N(m,n;f) + 2\pi \sum_{\substack{h \in \mathcal{B}_0(\Gamma_0(N),1) \\ t_h \in i[0,\theta]}} \frac{\sqrt{mn}\cdot \overline{\rho_h(\infty,m)}\rho_h(\infty,n)}{\cos(\pi|t_h|)} \int_{\frac{X}{2}}^{X} Y_{2|t_h|}(x) e^{i\alpha x} \frac{dx}{x} \\ &\lesssim \left(\frac{C}{T}\right)^{\frac{1}{2}} + (1+|\alpha|)X + \left(1 + \frac{T}{C}X^{-2\theta}\right)\left(1 + \frac{m^{\frac{1}{4}}(m,N)^{\frac{1}{4}} + n^{\frac{1}{4}}(n,N)^{\frac{1}{4}}}{N^{\frac{1}{2}}}\right. \\ &\qquad \left. + \min\left\{\frac{(mn)^{\frac{\theta}{2}+\frac{1}{8}}(mn,N)^{\frac{1}{8}}}{N^{\frac{1}{2}}}, \frac{(mn)^{\frac{1}{4}}(mn,N)^{\frac{1}{4}}}{N}\right\}\right) \end{aligned} \tag{4.15}$$

and for $|\alpha| < 1$ also

$$\begin{aligned} &\mathcal{M}_N(m,n;f) + 2\pi \sum_{\substack{h \in \mathcal{B}_0(\Gamma_0(N),1) \\ t_h \in i[0,\theta]}} \frac{\sqrt{mn}\cdot \overline{\rho_h(\infty,m)}\rho_h(\infty,n)}{\cos(\pi|t_h|)} \int_{\frac{X}{2}}^{X} Y_{2|t_h|}(x) e^{i\alpha x} \frac{dx}{x} \\ &\lesssim (1-|\alpha|)^{-\frac{1}{2}-\epsilon}\left[\left(\frac{C}{T}\right)^{\frac{1}{2}} + \left(1 + \frac{T}{C}X^{-2\theta}\right)\right. \\ &\quad \times \left(1 + \frac{m^{\frac{1}{8}}(m,N)^{\frac{1}{8}} + n^{\frac{1}{8}}(n,N)^{\frac{1}{8}}}{N^{\frac{1}{4}}} \min\left\{(mn)^{\frac{\theta}{2}}, \frac{m^{\frac{1}{8}}(m,N)^{\frac{1}{8}} + n^{\frac{1}{8}}(n,N)^{\frac{1}{8}}}{N^{\frac{1}{4}}}\right\}\right. \\ &\qquad \left.\left. + \min\left\{\frac{(mn)^{\frac{3\theta}{4}+\frac{1}{16}}(mn,N)^{\frac{1}{16}}}{N^{\frac{1}{4}}}, \frac{(mn)^{\frac{1}{4}}(mn,N)^{\frac{1}{4}}}{N}\right\}\right)\right]. \end{aligned} \tag{4.16}$$

We shall require the following proposition.

Proposition 4.3.1. *Let $A \ge 1$ and $n \in \mathbb{N}$. Then, we have*

$$\sum_{\substack{h \in \mathcal{B}_0(\Gamma_0(N),1) \\ |t_h| \le A}} \frac{n}{\cosh(\pi t_h)} |\rho_h(\infty,n)|^2 \ll_\epsilon A^2 + \frac{\sqrt{n}}{N}(n,N)^{\frac{1}{2}}(nN)^\epsilon.$$

Proof. For the full modular group, this is due to Kuznetsov [Kuz80, Eq. (5.19)] and only minor modifications yield the above, see for example [Top16, Lemma 2.9] or [GS12, Theorem 9]. □

Let us first prove (4.15). We split the summation over t_h in $\mathcal{M}_N(m,n;f)$ into various ranges $\mathcal{I}_1,\ldots,\mathcal{I}_4$, which are treated individually. They are

$$\begin{aligned}
\mathcal{I}_1 &= \left[0, \max\left\{1, X^{\frac{1}{2}}\right\}\right], \\
\mathcal{I}_2 &= \left[\tfrac{1}{12}||\alpha|^2-1|^{\frac{1}{2}}X, 2||\alpha|^2-1|^{\frac{1}{2}}X\right] \setminus \mathcal{I}_1, \\
\mathcal{I}_3 &= \left[\max\left\{1, X^{\frac{1}{2}}\right\}, \infty\right[\setminus \mathcal{I}_2, \\
\mathcal{I}_4 &= i\left[0, \frac{1}{2}\right].
\end{aligned}$$

The first way to treat the range $\mathcal{I}_1$ is to choose the basis (3.51) and use (3.53) as well as (3.54):

$$\sum_{\substack{h\in\mathcal{B}_0(\Gamma_0(N),1)\\ t_h\in\mathcal{I}_1}} \frac{\sqrt{mn}}{\cosh(\pi t_h)} \overline{\rho_h(\infty,m)}\rho_h(\infty,n)\widehat{f}(t_h)$$
$$\lesssim (mn)^{\theta} \sum_{M|N} \frac{1}{M} \sum_{\substack{h\in\mathcal{B}_0(\Gamma_0(N),1)\\ t_h\in\mathcal{I}_1\\ \text{new of level } M}} \sum_{d|\frac{N}{M}} (1+|t_h|)^{\epsilon} \sup_{t\in\mathcal{I}_1} |\widehat{f}(t)|.$$

Next, we use (4.5) to bound the transform and a uniform Weyl law to bound the number of Maass forms h of level M with $t_h \le T$ by $M^{1+\epsilon}T^2$ (see for example [Pal12, Corollary 3.2.3.]). We arrive at the bound

$$\lesssim (mn)^{\theta}\left(1+X^{\frac{1}{2}}\right). \tag{4.17}$$

A second way to treat the range $\mathcal{I}_1$ is to apply the Cauchy–Schwarz inequality in conjunction with Proposition 4.3.1 and (4.5):

$$\begin{aligned}
&\sum_{\substack{h\in\mathcal{B}_0(\Gamma_0(N),1)\\ t_h\in\mathcal{I}_1}} \frac{\sqrt{mn}}{\cosh(\pi t_h)}\overline{\rho_h(\infty,m)}\rho_h(\infty,n)\widehat{f}(t_h)\\
&\leq \left(\sum_{\substack{h\in\mathcal{B}_0(\Gamma_0(N),1)\\ t_h\in\mathcal{I}_1}} \frac{m}{\cosh(\pi t_h)}|\rho_h(\infty,m)|^2\right)^{\frac{1}{2}}\left(\sum_{\substack{h\in\mathcal{B}_0(\Gamma_0(N),1)\\ t_h\in\mathcal{I}_1}} \frac{n}{\cosh(\pi t_h)}|\rho_j(\infty,n)|^2\right)^{\frac{1}{2}}\sup_{t\in\mathcal{I}_1}|\widehat{f}(t)|\\
&\lesssim \left(1+X+\frac{\sqrt{m}}{N}(m,N)^{\frac{1}{2}}\right)^{\frac{1}{2}}\left(1+X+\frac{\sqrt{n}}{N}(n,N)^{\frac{1}{2}}\right)^{\frac{1}{2}}\frac{1}{1+X^{\frac{1}{2}}}\\
&\lesssim 1+X^{\frac{1}{2}}+\frac{m^{\frac{1}{4}}(m,N)^{\frac{1}{4}}+n^{\frac{1}{4}}(n,N)^{\frac{1}{4}}}{N^{\frac{1}{2}}}+\frac{(mn)^{\frac{1}{4}}(mn,N)^{\frac{1}{4}}}{N(1+X^{\frac{1}{2}})}.
\end{aligned} \tag{4.18}$$

We treat the range $\mathcal{I}_2$ in exactly the same manner and arrive at the inequalities

$$\begin{aligned}
\sum_{t_h\in\mathcal{I}_2}\frac{\sqrt{mn}}{\cosh(\pi t_h)}\overline{\rho_h(m)}\rho_h(n)\widehat{f}(t_h) &\lesssim (mn)^{\theta}\frac{\left(1+||\alpha|^2-1|^{\frac{1}{2}}X\right)^2}{1+||\alpha|^2-1|^{\frac{1}{2}}X}\\
&\lesssim (mn)^{\theta}\left(1+||\alpha|^2-1|^{\frac{1}{2}}X\right)
\end{aligned} \tag{4.19}$$

and

$$\begin{aligned}
&\sum_{\substack{h\in\mathcal{B}_0(\Gamma_0(N),1)\\ t_h\in\mathcal{I}_2}} \frac{\sqrt{mn}}{\cosh(\pi t_h)}\overline{\rho_h(\infty,m)}\rho_h(\infty,n)\widehat{f}(t_h)\\
&\lesssim \frac{\left(1+||\alpha|^2-1|^{\frac{1}{2}}X+\frac{m^{\frac{1}{4}}(m,N)^{\frac{1}{4}}}{N^{\frac{1}{2}}}\right)\left(1+||\alpha|^2-1|^{\frac{1}{2}}X+\frac{n^{\frac{1}{4}}(n,N)^{\frac{1}{4}}}{N^{\frac{1}{2}}}\right)}{1+X^{\frac{1}{2}}+||\alpha|^2-1|^{\frac{1}{2}}X}\\
&\lesssim 1+||\alpha|^2-1|^{\frac{1}{2}}X+\frac{m^{\frac{1}{4}}(m,N)^{\frac{1}{4}}+n^{\frac{1}{4}}(n,N)^{\frac{1}{4}}}{N^{\frac{1}{2}}}+\frac{(mn)^{\frac{1}{4}}(mn,N)^{\frac{1}{4}}}{N(1+||\alpha|^2-1|^{\frac{1}{2}}X)}.
\end{aligned} \tag{4.20}$$

We further split the range $\mathcal{I}_3$ into the dyadic ranges

$$\mathcal{I}_3(l)=[2^l\max\{1,X^{\frac{1}{2}}\},2^{l+1}\max\{1,X^{\frac{1}{2}}\}]\backslash\mathcal{I}_2,\quad l\geq 0.$$

Again, we can estimate

$$\sum_{\substack{h\in\mathcal{B}_0(\Gamma_0(N),1)\\ t_h\in\mathcal{I}_3(l)}}\frac{\sqrt{mn}}{\cosh(\pi t_h)}|\rho_h(\infty,m)\rho_h(\infty,n)|\lesssim (mn)^{\theta}2^{2l}(1+X) \tag{4.21}$$

and

$$\sum_{\substack{h\in\mathcal{B}_0(\Gamma_0(N),1)\\ t_h\in\mathcal{I}_3(l)}} \frac{\sqrt{mn}}{\cosh(\pi t_h)}|\rho_h(\infty,m)\rho_h(\infty,n)| \lesssim 2^{2l}(1+X)+2^l\left(1+X^{\frac{1}{2}}\right)\frac{m^{\frac{1}{4}}(m,N)^{\frac{1}{4}}+n^{\frac{1}{4}}(n,N)^{\frac{1}{4}}}{N^{\frac{1}{2}}}+\frac{(mn)^{\frac{1}{4}}(mn,N)^{\frac{1}{4}}}{N}, \quad (4.22)$$

However, this time we use (4.11) and (4.12) to deal with the transform. We have

$$\sup_{t\in\mathcal{I}_3(l)}|\widehat{f}(t)|\lesssim\begin{cases}\min\left\{\frac{1+X^{\frac{1}{2}}}{2^{2l}(1+X)},\frac{C}{T}\frac{1+X^{\frac{3}{2}}}{2^{4l}(1+X)^2}\right\}, & \text{for } l\le\log_2(\max\{1,X^{\frac{1}{2}}\}),\\ \min\left\{\frac{1}{2^{\frac{3}{2}l}(1+X)^{\frac{3}{4}}},\frac{C}{T}\frac{1}{2^{\frac{5}{2}l}(1+X)^{\frac{5}{4}}}\right\}, & \text{for } l>\log_2(\max\{1,X^{\frac{1}{2}}\}).\end{cases} \quad (4.23)$$

By combining (4.21), (4.22) and (4.23), we find that the contribution stemming from $l\le\log_2(\max\{1,X^{\frac{1}{2}}\})$ is

$$\begin{aligned}&\lesssim\sum_{l\le\log_2(\max\{1,X^{\frac{1}{2}}\})}\Bigg(1+X^{\frac{1}{2}}+2^{-l}\frac{m^{\frac{1}{4}}(m,N)^{\frac{1}{4}}+n^{\frac{1}{4}}(n,N)^{\frac{1}{4}}}{N^{\frac{1}{2}}}\\ &\qquad\qquad+\min\left\{(mn)^{\theta}(1+X)^{\frac{1}{2}},2^{-2l}\frac{(mn)^{\frac{1}{4}}(mn,N)^{\frac{1}{4}}}{N(1+X)^{\frac{1}{2}}}\right\}\Bigg)\\ &\lesssim 1+X^{\frac{1}{2}}+\frac{m^{\frac{1}{4}}(m,N)^{\frac{1}{4}}+n^{\frac{1}{4}}(n,N)^{\frac{1}{4}}}{N^{\frac{1}{2}}}+\min\left\{\frac{(mn)^{\frac{\theta}{2}+\frac{1}{8}}(mn,N)^{\frac{1}{8}}}{N^{\frac{1}{2}}},\frac{(mn)^{\frac{1}{4}}(mn,N)^{\frac{1}{4}}}{N}\right\},\end{aligned} \quad (4.24)$$

and the contribution from $l>\log_2(\max\{1,X^{\frac{1}{2}}\})$ is

$$\begin{aligned}&\lesssim\sum_{l>\log_2(\max\{1,X^{\frac{1}{2}}\})}\Bigg(\left(\frac{C}{T}\right)^{\frac{1}{2}+\delta}2^{-\delta l}+2^{-\frac{l}{2}}\frac{m^{\frac{1}{4}}(m,N)^{\frac{1}{4}}+n^{\frac{1}{4}}(n,N)^{\frac{1}{4}}}{N^{\frac{1}{2}}}\\ &\qquad\qquad+2^{-\frac{l}{2}}\min\left\{\frac{(mn)^{\frac{\theta}{2}+\frac{1}{8}}(mn,N)^{\frac{1}{8}}}{N^{\frac{1}{2}}},\frac{(mn)^{\frac{1}{4}}(mn,N)^{\frac{1}{4}}}{N}\right\}\Bigg)\\ &\lesssim\left(\frac{C}{T}\right)^{\frac{1}{2}}+\frac{m^{\frac{1}{4}}(m,N)^{\frac{1}{4}}+n^{\frac{1}{4}}(n,N)^{\frac{1}{4}}}{N^{\frac{1}{2}}}+\min\left\{\frac{(mn)^{\frac{\theta}{2}+\frac{1}{8}}(mn,N)^{\frac{1}{8}}}{N^{\frac{1}{2}}},\frac{(mn)^{\frac{1}{4}}(mn,N)^{\frac{1}{4}}}{N}\right\},\end{aligned} \quad (4.25)$$

for a sufficiently small $\delta>0$.

For the contribution from $\mathcal{I}_4$, we first note that we have $|t_h| \leq \theta$ for $t_h \in \mathcal{I}_4$ by [Kim03]. We insert (4.6) and find further that

$$4\pi \sum_{\substack{h \in \mathcal{B}_0(\Gamma_0(N),1) \\ t_h \in i[0,\theta]}} \frac{\sqrt{mn}}{\cosh(\pi t_h)} \overline{\rho_h(\infty,m)} \rho_h(\infty,n) \left(-\frac{1}{2} \int_{\frac{X}{2}}^{X} Y_{2|t_h|}(x) e^{i\alpha x} \frac{dx}{x} + O_\epsilon \left(1 + \frac{T}{C} X^{-2|t_h|-\epsilon} \right) \right)$$

$$= -2\pi \sum_{\substack{h \in \mathcal{B}_0(\Gamma_0(N),1) \\ t_h \in i[0,\theta]}} \frac{\sqrt{mn} \cdot \overline{\rho_h(\infty,m)} \rho_h(\infty,n)}{\cos(\pi|t_h|)} \int_{\frac{X}{2}}^{X} Y_{2|t_h|}(x) e^{i\alpha x} \frac{dx}{x} +$$

$$O_\epsilon \left(\left(1 + \frac{T}{C} X^{-2\theta-\epsilon} \right) \min \left\{ (mn)^\theta, 1 + \frac{m^{\frac{1}{4}}(m,N)^{\frac{1}{4}} + n^{\frac{1}{4}}(n,N)^{\frac{1}{4}}}{N^{\frac{1}{2}}} + \frac{(mn)^{\frac{1}{4}}(mn,N)^{\frac{1}{4}}}{N} \right\} \right). \tag{4.26}$$

Combining the minimum of (4.17) and (4.18), the minimum of (4.19) and (4.20), (4.24), (4.25), and (4.26) gives (4.15).

Let us now turn our attention to (4.16). This time, we split up into the intervals

$$\mathcal{I}_1 = [0,1],$$
$$\mathcal{I}_2 = [1,\infty[,$$
$$\mathcal{I}_3 = i\left[0, \frac{1}{2}\right].$$

By making use of (4.5), we find that the contribution from $\mathcal{I}_1$ is bounded by

$$\lesssim \min \left\{ (mn)^\theta, 1 + \frac{m^{\frac{1}{4}}(m,N)^{\frac{1}{4}} + n^{\frac{1}{4}}(n,N)^{\frac{1}{4}}}{N^{\frac{1}{2}}} + \frac{(mn)^{\frac{1}{4}}(mn,N)^{\frac{1}{4}}}{N} \right\}. \tag{4.27}$$

As before, we split up $\mathcal{I}_2$ into dyadic ranges $\mathcal{I}_2(l) = [2^l, 2^{l+1}]$, $l \geq 0$ and use

$$\sup_{t \in \mathcal{I}_2(l)} |\widehat{f}(t)| \lesssim \min \left\{ (1-|\alpha|)^{-\frac{1}{4}} 2^{-\frac{3}{2}l}, \frac{C}{T}(1-|\alpha|)^{-\frac{3}{4}} 2^{-\frac{5}{2}l} \right\},$$

which follows from (4.11) and (4.12). Hence, we find that the contribution from $\mathcal{I}_2$ is bounded by

$$\lesssim (1-|\alpha|)^{-\frac{1}{2}-\frac{\delta}{2}} \sum_{l \geq 0} \left(\left(\frac{C}{T} \right)^{\frac{1}{2}+\delta} 2^{-\delta l} + \right.$$

$$\min \left\{ (mn)^{\frac{\theta}{2}-\theta\delta} \frac{m^{\frac{1}{8}+\frac{\delta}{4}}(m,N)^{\frac{1}{8}+\frac{\delta}{4}} + n^{\frac{1}{8}+\frac{\delta}{4}}(n,N)^{\frac{1}{8}+\frac{\delta}{4}}}{N^{\frac{1}{4}+\frac{\delta}{2}}} 2^{-\delta l}, \frac{m^{\frac{1}{4}}(m,N)^{\frac{1}{4}} + n^{\frac{1}{4}}(n,N)^{\frac{1}{4}}}{N^{\frac{1}{2}}} 2^{-\frac{1}{2}l} \right\}$$

$$\left. + \min \left\{ (mn)^{\frac{3\theta}{4}-\theta\delta} \frac{(mn)^{\frac{1}{16}+\frac{\delta}{4}}(mn,N)^{\frac{1}{16}+\frac{\delta}{4}}}{N^{\frac{1}{4}+\delta}} 2^{-2\delta l}, \frac{(mn)^{\frac{1}{4}}(mn,N)^{\frac{1}{4}}}{N} 2^{-\frac{3}{2}l} \right\} \right),$$

which is

$$\lesssim (1-|\alpha|)^{-\frac{1}{2}-\epsilon}\left(\frac{m^{\frac{1}{8}}(m,N)^{\frac{1}{8}}+n^{\frac{1}{8}}(n,N)^{\frac{1}{8}}}{N^{\frac{1}{4}}}\min\left\{(mn)^{\frac{\theta}{2}},\frac{m^{\frac{1}{8}}(m,N)^{\frac{1}{8}}+n^{\frac{1}{8}}(n,N)^{\frac{1}{8}}}{N^{\frac{1}{4}}}\right\}\right.$$
$$\left.+\left(\frac{C}{T}\right)^{\frac{1}{2}}+\min\left\{\frac{(mn)^{\frac{3\theta}{4}+\frac{1}{16}}(mn,N)^{\frac{1}{16}}}{N^{\frac{1}{4}}},\frac{(mn)^{\frac{1}{4}}(mn,N)^{\frac{1}{4}}}{N}\right\}\right) \quad (4.28)$$

for $\delta > 0$ small enough. The contribution from $\mathcal{I}_3$ is the same as in (4.26). Combining (4.27), (4.28), and (4.26) gives (4.16).

4.4 PUTTING EVERYTHING TOGETHER

In order to show Theorem 4.0.4, we add up all the inequalities (4.4), (4.13), (4.14), (4.15) respectively (4.16), and make the choice $T = O(N^{\frac{2}{3}}C^{\frac{2}{3}})$, which is allowed since $N \ll \min\{(mn)^{\frac{1}{4}}, C^{\frac{1}{2}}\}$. One may note that we have

$$\frac{(m,n)^{\frac{1}{2}}}{\sqrt{C}} \leq X^{\frac{1}{2}} \leq 1+X$$

and

$$\frac{T}{C}X^{-2\theta} \ll N^{\frac{2}{3}-4\theta}C^{2\theta-\frac{1}{3}}\cdot N^{4\theta}(mn)^{-\theta} \ll 1.$$

Theorem 4.0.1 follows at once by estimating the range $c \leq (1+|\alpha|^{\frac{2}{3}})N^{\frac{2}{3}}(mn)^{\frac{1}{3}}$ trivially using the Weil bound, which gives

$$\sum_{\substack{c\leq(1+|\alpha|^{\frac{2}{3}})N^{\frac{2}{3}}(mn)^{\frac{1}{3}}\\ c\equiv 0 \bmod (N)}}\frac{|S(m,n;c)|}{c} \lesssim \frac{\left((1+|\alpha|^{\frac{2}{3}})N^{\frac{2}{3}}(mn)^{\frac{1}{3}}\right)^{\frac{1}{2}}}{N} \lesssim (1+|\alpha|^{\frac{1}{3}})\frac{(mn)^{\frac{1}{6}}}{N^{\frac{2}{3}}}.$$

For the remaining range $(1+|\alpha|^{\frac{2}{3}})N^{\frac{2}{3}}(mn)^{\frac{1}{6}} \leq c \leq C$, we use Theorem 4.0.4. Furthermore, note that

$$\int_1^\infty |Y_{2t}(x)|\frac{dx}{x} \ll \int_1^\infty x^{-\frac{3}{2}}dx \ll 1$$

uniformly for $t \leq \theta$ and hence we have

$$\sum_{\substack{h\in\mathcal{B}_0(\Gamma_0(N),1)\\ t_h\in i[0,\theta]}}\frac{\sqrt{mn}\cdot\overline{\rho_h(\infty,m)}\rho_h(\infty,n)}{\cos(\pi|t_h|)}\int_1^\infty Y_{2|t_h|}(x)e^{i\alpha x}\frac{dx}{x}$$
$$\lesssim \min\left\{(mn)^{\theta}, 1+\frac{m^{\frac{1}{4}}(m,N)^{\frac{1}{4}}+n^{\frac{1}{4}}(n,N)^{\frac{1}{4}}}{N^{\frac{1}{2}}}+\frac{(mn)^{\frac{1}{4}}(mn,N)^{\frac{1}{4}}}{N}\right\}.$$

This proves Theorem 4.0.1. In order to prove Corollary 4.0.2, we need to show

$$\sum_{\substack{h\in\mathcal{B}_0(\Gamma_0(N),1)\\ t_h\in i[0,\theta]}} \frac{\sqrt{mn}\cdot\overline{\rho_h(\infty,m)}\rho_h(\infty,n)}{\cos(\pi|t_h|)}\int_X^1 Y_{2|t_h|}(x)e^{i\alpha x}\frac{dx}{x}\lesssim C^{2\theta},$$

when $C\geq\sqrt{mn}$. This follows from the two estimates

$$\int_X^1 |Y_{2t}(x)|\frac{dx}{x}\ll_\epsilon\int_X^1 x^{-2\theta-1-\epsilon}dx\ll_\epsilon X^{-2\theta-\epsilon}$$

and

$$\sum_{\substack{h\in\mathcal{B}_0(\Gamma_0(N),1)\\ t_h\in i[0,\theta]}} \frac{\sqrt{mn}\cdot|\rho_h(\infty,m)\rho_h(\infty,n)|}{\cos(\pi|t_h|)}\ll_\epsilon (mn)^{\theta+\epsilon}.$$

Theorem 4.0.3 is proved analogously.

4.5 TRANSFORM ESTIMATES

In this section, we prove the claimed upper bounds in Lemma 4.0.6 on the transforms of f. Since all the estimates are very different in nature, we split them up into multiple lemmata. We generally follow the arguments of [ST09] and [DI83], but tweak them to account for our introduced twist. First, we shall need two preliminary lemmata, which will be used frequently.

Lemma 4.5.1. *Let $F,G\in C([A,B],\mathbb{C})$ with G having a continuous derivative. Then,*

$$\left|\int_A^B F(x)G(x)dx\right|\ll(\|G\|_\infty+\|G'\|_1)\sup_{C\in[A,B]}\left|\int_A^C F(x)dx\right|.$$

Proof. We integrate by parts and find

$$\int_A^B F(x)G(x)dx=\int_A^B F(x)dx\cdot G(B)-\int_A^B\int_A^y F(x)dx\cdot G'(y)dy,$$

from which the first statement is trivially deduced. □

Lemma 4.5.2. *Let $G,H\in C^1([A,B],\mathbb{C})$ and assume G has a zero and H' has at most K zeros. Then, we have*

$$\|GH\|_\infty+\|(GH)'\|_1\ll_K\|G'\|_1\|H\|_\infty.$$

Proof. We have $\|GH\|_\infty \le \|G\|_\infty \|H\|_\infty$ and $\|G\|_\infty \le \|G'\|_1$ since we have $G(b) = \int_a^b G'(x)dx$, where a is a zero of G. Furthermore, we have

$$\|(GH)'\|_1 \le \|G'H\|_1 + \|GH'\|_1 \le \|G'\|_1\|H\|_\infty + \|G\|_\infty\|H'\|_1 \le \|G'\|_1(\|H\|_\infty + \|H'\|_1)$$

and

$$\|H'\|_1 \le 2(K+1)\|H\|_\infty$$

by splitting up the integral into intervals on which H' has a constant sign. □

Lemma 4.5.3. *Let f be as in (4.2) and $|\alpha| \le 1$. Then, we have*

$$\widetilde{f}(t) \ll \frac{1 + |\log(X)|}{1 + X^{\frac{1}{2}} + |1 - |\alpha|^2|^{\frac{1}{2}} X}, \qquad \forall t \in \mathbb{R}.$$

Proof. We follow the proof of [DI83, Lemma 7.1] and [ST09, Prop. 5]. To prove the first statement we use the Bessel representation (A.11)

$$J_t(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{i(x\sin\xi - t\xi)} d\xi.$$

Integration by parts yields

$$\begin{aligned}\int_0^\infty e^{ix\sin\xi}\frac{f(x)}{x}dx &= \int_0^\infty e^{ix(\sin\xi+\alpha)}\frac{g(x)}{x}dx \\ &= \frac{i}{\sin\xi+\alpha}\int_0^\infty e^{ix(\sin\xi+\alpha)}\left(\frac{g(x)}{x}\right)' dx \\ &\ll \min\left\{1, X^{-1}|\sin\xi+\alpha|^{-1}\right\}.\end{aligned}$$

Hence, we find

$$\widetilde{f}(t) \ll \int_0^{2\pi} \min\left\{1, X^{-1}|\sin\xi+\alpha|^{-1}\right\} d\xi.$$

Now, clearly $\widetilde{f}(t) \ll 1$. For $X \ge 1$, we can do better though. We have $|\sin\xi + \alpha| \ge ||\sin\xi| - |\alpha||$. Thus, we may assume $\xi \in [0, \frac{\pi}{2}]$ and $\alpha \ge 0$. Set $\alpha = \sin\phi$ with $\phi \in [0, \frac{\pi}{2}]$. Then, we have

$$\sin\xi - \alpha = 2\sin\left(\frac{\xi-\phi}{2}\right)\sin\left(\frac{\pi-\xi-\phi}{2}\right).$$

Now, for $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ we have $|\sin(x)| \asymp |x|$. Thus,

$$\begin{aligned}\widetilde{f}(t) &\ll \int_0^{\frac{\pi}{2}} \min\left\{1, X^{-1}|\xi-\phi|^{-1}|\pi-\xi-\phi|^{-1}\right\} d\xi \\ &\ll \int_0^{\frac{\pi}{2}} \min\left\{1, X^{-1}|\xi-\phi|^{-1}|\tfrac{\pi}{2}-\phi|^{-1}, X^{-1}|\xi-\phi|^{-2}\right\} d\xi \\ &\ll \min\left\{\frac{1+\log(X)}{|\frac{\pi}{2}-\phi|X}, X^{-\frac{1}{2}}\right\}.\end{aligned}$$

Now, we just have to note that $\frac{\pi}{2} - \phi \asymp \sin(\frac{\pi}{2} - \phi) = \sqrt{1-|\alpha|^2}$. □

Lemma 4.5.4. *Let f be as in (4.2) and $|\alpha| \geq 1$. Then, we have*

$$\widetilde{f}(t) \ll \frac{1 + |\log(X)|}{1 + X^{\frac{1}{2}} + ||\alpha|^2 - 1|^{\frac{1}{2}} X}, \quad \forall t \in \mathbb{R}.$$

Proof. As before, we find $\widetilde{f}(t) \ll 1$ and for $X \geq 1$ we have

$$\begin{aligned}
\widetilde{f}(t) &\ll \int_0^{\frac{\pi}{2}} \min\left\{1, X^{-1}(|\alpha| - |\sin\xi|)^{-1}\right\} d\xi \\
&\ll \int_0^{\frac{\pi}{2}} \min\left\{1, X^{-1}(|\alpha| - 1 + \tfrac{1}{\pi}(\tfrac{\pi}{2} - \xi)^2)^{-1}\right\} d\xi \\
&\ll \int_0^{\frac{\pi}{2}} \min\left\{1, X^{-1}(|\alpha| - 1)^{-1}, X^{-1}(|\alpha| - 1)^{-\frac{1}{2}}(\tfrac{\pi}{2} - \xi)^{-1}, X^{-1}(\tfrac{\pi}{2} - \xi)^2\right\} d\xi \\
&\ll \min\left\{\frac{1}{||\alpha| - 1|X}, \frac{1 + \log(X)}{||\alpha| - 1|^{\frac{1}{2}} X}, X^{-\frac{1}{2}}\right\}.
\end{aligned}$$

□

We also require some more refined estimates. For this, we consider the different regions of the J-Bessel function.

Lemma 4.5.5. *Let f as in 4.2 and $|\alpha| \leq 1$. Then, we have for $t \geq 8$*

$$\begin{aligned}
\int_0^{\frac{t}{2}} J_t(y) f(y) \frac{dy}{y} &\ll \mathbb{1}_{[2X/3,\infty[}(t) \cdot t^{-\frac{1}{2}} e^{-\frac{2}{5}t}, \\
\int_{\frac{t}{2}}^{t - t^{\frac{1}{3}}} J_t(y) f(y) \frac{dy}{y} &\ll \mathbb{1}_{[\frac{1}{4},\infty[}(X) \mathbb{1}_{[X/3,4X]}(t) \cdot t^{-1} (\log(t))^{\frac{2}{3}}, \\
\int_{t - t^{\frac{1}{3}}}^{t + t^{\frac{1}{3}}} J_t(y) f(y) \frac{dy}{y} &\ll \mathbb{1}_{[\frac{1}{4},\infty[}(X) \mathbb{1}_{[3X/16,3X]}(t) \cdot t^{-1}, \\
\int_{t + t^{\frac{1}{3}}}^{\infty} J_t(y) f(y) \frac{dy}{y} &\ll \mathbb{1}_{[\frac{1}{4},\infty[}(X) \mathbb{1}_{[0,3X/2]}(t) \cdot t^{-1} \min\left\{|1 - |\alpha||^{-\frac{1}{4}}, \left(\frac{X}{t}\right)^{\frac{1}{2}}\right\}, \qquad (4.29)
\end{aligned}$$

where $\mathbb{1}_{\mathcal{I}}$ is the characteristic function of the interval $\mathcal{I}$.

Proof. We require some uniform estimates on the J-Bessel functions of real order. For small argument, we have exponential decay:

$$0 \leq J_t(x) \leq \frac{e^{-tF(0,x/t)}}{(1 - (x/t)^2)^{\frac{1}{4}} \sqrt{2\pi t}}, \quad \forall x < t, \qquad (4.30)$$

where $F(0, x) = \log\left(\frac{1+\sqrt{1-x^2}}{x}\right) - \sqrt{1 - x^2}$. The left-hand side follows from the fact that the first zero of the Bessel function of order t is $> t$ and the right-hand side follows from [Wat44, pp. 252-255]. We will also make use of Langer's formulae see [EMOT81, pp. 30,89]. The first formula is

$$J_t(x) = w^{-\frac{1}{2}}(w - \arctan(x))^{\frac{1}{2}} \left(\frac{\sqrt{3}}{2} J_{\frac{1}{3}}(z) - \frac{1}{2} Y_{\frac{1}{3}}(z)\right) + O(t^{-\frac{4}{3}}), \quad \forall x > t, \qquad (4.31)$$

where

$$w = \sqrt{\frac{x^2}{t^2} - 1} \text{ and } z = t(w - \arctan(w)).$$

The second one is

$$J_t(x) = \frac{1}{\pi} w^{-\frac{1}{2}} (\operatorname{artanh}(w) - w)^{\frac{1}{2}} K_{\frac{1}{3}}(z) + O(t^{-\frac{4}{3}}), \quad \forall x < t, \tag{4.32}$$

where

$$w = \sqrt{1 - \frac{x^2}{t^2}} \text{ and } z = t(\operatorname{artanh}(w) - w).$$

And finally, for the transitional range $|x - t| \leq t^{\frac{1}{3}}$, we have

$$J_t(x) \ll t^{-\frac{1}{3}}, \tag{4.33}$$

by [Wat44, pp. 244-247].

The first inequality follows directly from (4.30)

$$\int_0^{\frac{t}{2}} J_t(y) f(y) \frac{dy}{y} \ll t^{-\frac{1}{2}} e^{-\frac{2}{5}t} \cdot \frac{X}{X}.$$

Note, that if $X \leq \frac{1}{2}$, then this covers everything. Thus, we may assume $X \geq \frac{1}{2}$ from now on. For the range $[\frac{t}{2}, t - t^{\frac{1}{3}}]$, we use (4.32) and $z^{\frac{1}{2}} K_{\frac{1}{3}}(z) \ll e^{-z}$, $\forall z \geq 0$. Thus, we find

$$J_t(y) \ll (t^2 - y^2)^{-\frac{1}{4}} e^{-z} + O(t^{-\frac{4}{3}}).$$

Now, if $y \leq \min\{t - 9t^{\frac{1}{3}}(\log t)^{\frac{2}{3}}, t - t^{\frac{1}{3}}\}$ we have $z \geq \log t$ and thus $J_t(y) \ll t^{-\frac{4}{3}}$, otherwise we have $J_t(y) \ll t^{-\frac{1}{3}}$. We conclude

$$\int_{\frac{t}{2}}^{t - t^{\frac{1}{3}}} J_t(y) f(y) \frac{dy}{y} \ll t^{-\frac{4}{3}} \cdot \frac{X}{X} + t^{-\frac{1}{3}} \cdot \frac{t^{\frac{1}{3}} (\log(t))^{\frac{2}{3}}}{t}.$$

For the range $t - t^{\frac{1}{3}} \leq y \leq t + t^{\frac{1}{3}}$, we use (4.33) and get

$$\int_{t - t^{\frac{1}{3}}}^{t + t^{\frac{1}{3}}} J_t(y) f(y) \frac{dy}{y} \ll t^{-\frac{1}{3}} \cdot \frac{t^{\frac{1}{3}}}{t}.$$

We are left to deal with the range $t + t^{\frac{1}{3}} \leq y$. We make a change of variable $y \to ty$ and we are left to estimate

$$\int_{1 + t^{-\frac{2}{3}}}^{\infty} J_t(ty) e^{i\alpha t y} g(ty) \frac{dy}{y}. \tag{4.34}$$

We make use of (4.31) and find $z \gg 1$ in this range of y. By making use of Langer's formula (4.31), we introduce an error of the size

$$\ll t^{-\frac{4}{3}} \cdot \frac{X}{X},$$

which is sufficient. Since $z \gg 1$, we are able to make use of the classical estimates (A.18), (A.19):

$$\begin{aligned} J_{\frac{1}{3}}(z) &= \sqrt{\frac{2}{\pi z}}\left(\cos(z - \tfrac{\pi}{6} - \tfrac{\pi}{4}) + O(z^{-1})\right), \\ Y_{\frac{1}{3}}(z) &= \sqrt{\frac{2}{\pi z}}\left(\sin(z - \tfrac{\pi}{6} - \tfrac{\pi}{4}) + O(z^{-1})\right). \end{aligned} \tag{4.35}$$

Inserting (4.35) into (4.34) introduces another error of the size

$$t^{-\frac{1}{2}} \int_{1+t^{-\frac{2}{3}}}^{\infty} w^{-\frac{1}{2}} z^{-1} g(ty) \frac{dy}{y},$$

where $w = \sqrt{y^2 - 1}$ and $z = t(w - \arctan(w))$. We have $z \gg t \min\{w^3, w\}$ and thus we are able to estimate the above as

$$\begin{aligned} &\ll t^{-\frac{3}{2}} \int_{1+t^{-\frac{2}{3}}}^{2} \frac{g(ty)}{(y^2-1)^{\frac{7}{4}} y} dy + t^{-\frac{3}{2}} \int_{2}^{\infty} \frac{g(ty)}{(y^2-1)^{\frac{3}{4}} y} dy \\ &\ll t^{-\frac{3}{2}} \int_{1+t^{-\frac{2}{3}}}^{2} \frac{g(ty) y}{(y^2-1)^{\frac{7}{4}}} dy + t^{-\frac{3}{2}} \int_{2}^{\infty} \frac{g(ty)}{y^{\frac{5}{2}}} dy \\ &\ll \|g'\|_1 \cdot t^{-1} + t^{-\frac{3}{2}}, \end{aligned}$$

where we have made use of Lemmata 4.5.1 and 4.5.2 with $F(y) = y(y^2-1)^{-\frac{7}{4}}$ and $G(y) = g(ty)$, respectively $F(y) = y^{-\frac{5}{2}}$ and $G(y) = g(ty)$. This is again sufficient. For the main term, we have to consider

$$t^{-\frac{1}{2}} \int_{1+t^{-\frac{2}{3}}}^{\infty} e^{it(\pm\omega(y)+\alpha y)} \frac{g(ty)}{(y^2-1)^{\frac{1}{4}} y} dy, \tag{4.36}$$

where

$$\begin{aligned} \omega(y) &= \sqrt{y^2-1} - \arctan\sqrt{y^2-1}, \\ \omega'(y) &= \frac{\sqrt{y^2-1}}{y}. \end{aligned}$$

We would like to integrate $t(\pm\omega'(y)+\alpha)e^{it(\pm\omega(y)+\alpha y)}$ by parts, but for the sign '$-\mathrm{sign}(\alpha)$' and $y_0 = (1-\alpha^2)^{-\frac{1}{2}}$, we have $\omega'(y_0) = |\alpha|$ and we pick up a stationary phase. Let us first assume α is close to 0, such that $y_0 < 1 + t^{-\frac{2}{3}}$. For $|\alpha| \ll t^{-\frac{1}{3}}$ or the sign '$\mathrm{sign}(\alpha)$', we have $|\pm\omega'(1+t^{-\frac{2}{3}}) + \alpha| \gg t^{-\frac{1}{3}}$ and we get by means of Lemmata 4.5.1 and 4.5.2 with $F(y) = (\pm\omega'(y)+\alpha)e^{it(\pm\omega(y)+\alpha y)}$, $G(y) = g(ty)$ and $H(y) = [(\pm\omega'(y)+\alpha)(y^2-1)^{\frac{1}{4}} y]^{-1}$ a satisfying contribution of t^{-1}. So, from now on, we can assume $\alpha > 0$, $\alpha \geq kt^{-\frac{1}{3}}$, for some small constant k, and the sign being '$-$'. We treat first the case $\alpha < 1$, where we make use of a Taylor expansion around y_0. We split up the integral (4.36) into three parts $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3$ corresponding to the intervals $[1+t^{-\frac{2}{3}}, y_0 - A]$, $[y_0 - A, y_0 + A]$, $[y_0 + A, \infty[$, respectively. For $\mathcal{I}_1$ and $\mathcal{I}_3$, we again make use of Lemmata 4.5.1 and 4.5.2 with $F(y) =$

$(\omega'(y)-\alpha)e^{it(\omega(y)-\alpha y)}, G(y) = g(ty)$, and $H(y) = [(\omega'(y)-\alpha)(y^2-1)^{\frac{1}{4}}y]^{-1}$. Thus, we need lower bounds on

$$R(x) = \sqrt{x^2-1} - \alpha x \text{ and } (x^2-1)^{\frac{1}{4}}.$$

We have

$$R'(x) = \frac{x}{\sqrt{x^2-1}} - \alpha \text{ and } R''(x) = -\frac{1}{(x^2-1)^{\frac{3}{2}}}.$$

We have that $R'(x)$ is decreasing and positive and hence $R(x)$ is increasing with a zero at y_0. Furthermore, we have $R''(x)$ is increasing and negative. We conclude

$$\begin{aligned} R(y_0+A) &\geq R(y_0) + F'(y_0)\cdot A + R''(y_0)\cdot \frac{A^2}{2} \\ &= \frac{1-\alpha^2}{\alpha}\cdot A - \left(\frac{1-\alpha^2}{\alpha^2}\right)^{\frac{3}{2}} \cdot \frac{A^2}{2} \\ &= \frac{1-\alpha^2}{\alpha}\cdot A \cdot \left(1 - \frac{(1-\alpha^2)^{\frac{1}{2}}}{\alpha^2}\cdot\frac{A}{2}\right) \\ &\gg \frac{1-\alpha^2}{\alpha}\cdot A, \end{aligned}$$

for $A \leq \alpha^2(1-\alpha^2)^{-\frac{1}{2}}$. We also have

$$\begin{aligned} -R(y_0-A) &\geq -R(y_0) + R'(y_0)A \\ &\gg \frac{1-\alpha^2}{\alpha}\cdot A. \end{aligned}$$

For the second factor, we have

$$((y_0+A)^2-1)^{\frac{1}{4}} \geq \left(\frac{\alpha^2}{1-\alpha^2}\right)^{\frac{1}{4}}$$

and

$$((y_0-A)^2-1)^{\frac{1}{4}} \geq \left(\frac{\alpha^2}{1-\alpha^2} - \frac{2A}{(1-\alpha^2)^{\frac{1}{2}}}\right)^{\frac{1}{4}} \gg \left(\frac{\alpha^2}{1-\alpha^2}\right)^{\frac{1}{4}}$$

for $A \leq \frac{1}{4}\alpha^2(1-\alpha^2)^{-\frac{1}{2}}$. Thus, for $A \leq \frac{1}{4}\alpha^2(1-\alpha^2)^{-\frac{1}{2}}$, we find that the contribution from $\mathcal{I}_3$ is at most

$$t^{-\frac{3}{2}}\frac{1}{\left(\frac{1-\alpha^2}{\alpha}\right)A\cdot\left(\frac{\alpha^2}{1-\alpha^2}\right)^{\frac{1}{4}}} \ll t^{-\frac{3}{2}}\frac{\alpha^{\frac{1}{2}}}{(1-\alpha^2)^{\frac{3}{4}}A}.$$

We claim that $-R(x)(x^2-1)^{\frac{1}{4}}$ increases first and then decreases in $[1, y_0]$. For this, it suffices to prove that its derivative has exactly one zero in that interval and is positive at $1+\epsilon$. Note, that since our function is zero at the endpoints, we have by Rolle's Theorem that there is at least a zero of the derivative. The derivative is

$$\frac{3\alpha x^2 - 3x(x^2-1)^{\frac{1}{2}} - 2\alpha}{2(x^2-1)^{\frac{3}{4}}},$$

which is clearly positive at $1+\epsilon$. Assume now that we have two zeros y_1, y_2 in $[1, y_0]$. They both satisfy the equation

$$3\alpha x^2 - 3x(x^2-1)^{\frac{1}{2}} - 2\alpha = 0 \Rightarrow 9(1-\alpha^2)x^4 + (12\alpha^2-9)x^2 - 4\alpha^2 = 0.$$

Now, by Vieta's formula we have

$$2 \le y_1^2 + y_2^2 = \frac{9-12\alpha^2}{9(1-\alpha^2)} = \frac{4}{3} - \frac{1}{3(1-\alpha^2)} \le \frac{4}{3}$$

and thus a contradiction. With this information, we conclude that if $\alpha \ge Kt^{-\frac{1}{3}}$, for some large constant K, we have that the contribution from $\mathcal{I}_1$ is at most

$$\max\left\{t^{-1}, t^{-\frac{3}{2}}\frac{\alpha^{\frac{1}{2}}}{(1-\alpha^2)^{\frac{3}{4}}A}\right\}.$$

Furthermore, we estimate the integral over $\mathcal{I}_2$ trivially and get the bound

$$t^{-\frac{1}{2}}A\frac{(1-\alpha^2)^{\frac{3}{4}}}{\alpha^{\frac{1}{2}}}.$$

By choosing $A = t^{-\frac{1}{2}}\alpha^{\frac{1}{2}}(1-\alpha^2)^{-\frac{1}{2}}$, which we are allowed for K large enough, we get that (4.36) is bounded by

$$t^{-1}(1-|\alpha|)^{-\frac{1}{4}}.$$

We are left to deal with the case $\alpha \asymp t^{-\frac{1}{3}}$. In this case, we elongate the interval $\mathcal{I}_2$ to $[1+t^{-\frac{2}{3}}, y_0 + A]$ and estimate trivially again. By setting $A = \frac{1}{4}\alpha^2(1-\alpha^2)^{-\frac{1}{2}}$, we find that in this case one also has a bound of t^{-1} for $\mathcal{I}_2, \mathcal{I}_3$. This proves the first half of (4.29).

Let us assume now that $\alpha \ge \frac{2\sqrt{2}}{3}$ such that α is close to 1 and $y_0 \ge 3$. Assume $2X/t \le \frac{y_0}{2}$. In this case, the integral over $\mathcal{I}_2$ and $\mathcal{I}_3$ are 0, furthermore we have

$$\begin{aligned}\min_{\substack{x\in[1+t^{-\frac{2}{3}}, y_0/2] \\ x\in\frac{1}{t}\operatorname{Supp} g}} -R(x)(x^2-1)^{\frac{1}{4}} &= \min_{\substack{x\in[1+t^{-\frac{2}{3}}, y_0/2] \\ x\in\frac{1}{t}\operatorname{Supp} g}} \frac{1-(1-\alpha^2)x^2}{\alpha x + \sqrt{x^2-1}}(x^2-1)^{\frac{1}{4}} \\ &\gg \min\left\{t^{-\frac{1}{6}}, \left(\frac{X}{t}\right)^{-\frac{1}{2}}\right\},\end{aligned}$$

thus the contribution from $\mathcal{I}_1$ is bounded by

$$t^{-\frac{3}{2}}\left(t^{\frac{1}{6}} + \left(\frac{X}{t}\right)^{\frac{1}{2}}\right).$$

Similarly, for $\frac{1}{3}X/t \ge 2y_0$, we have that the integral over $\mathcal{I}_1$ and $\mathcal{I}_2$ are 0, and furthermore

$$\begin{aligned}\min_{\substack{x\in[2y_0,\infty[\\ x\in\frac{1}{t}\operatorname{Supp} g}} R(x)(x^2-1)^{\frac{1}{4}} &= \min_{\substack{x\in[2y_0,\infty[\\ x\in\frac{1}{t}\operatorname{Supp} g}} \frac{(1-\alpha^2)x^2-1}{\alpha x + \sqrt{x^2-1}}(x^2-1)^{\frac{1}{4}} \\ &\gg \left(\frac{X}{t}\right)^{-\frac{1}{2}},\end{aligned}$$

hence the contribution from $\mathcal{I}_3$ is bounded by

$$t^{-\frac{3}{2}}\left(\frac{X}{t}\right)^{\frac{1}{2}}.$$

Finally, when $X/t \asymp y_0$ we are able to replace $|1-|\alpha||^{-\frac{1}{4}}$ by $(X/t)^{\frac{1}{2}}$, which proves the last inequality in full for $|\alpha| < 1$.

Now, let us have a look at $\alpha = 1$. We proceed as before only that this time the stationary phase is at infinity. Thus, we can directly apply Lemmata 4.5.1 and 4.5.2 with $F(y) = (\omega'(y)-1)e^{it(\omega(y)-1y)}$, $G(y) = g(ty)$, and $H(y) = [(\omega'(y)-1)(y^2-1)^{\frac{1}{4}}y]^{-1}$. We need an upper bound on the quantity

$$\frac{1}{(y-\sqrt{y^2-1})(y^2-1)^{\frac{1}{4}}} \text{ for } y \in [1+t^{-\frac{2}{3}}, \infty[\text{ and } ty \in \operatorname{Supp} g.$$

This function decreases and then increases. Thus, it takes its maximum at the boundary. The values at the boundary are easily bounded by

$$\max\left\{t^{\frac{1}{6}}, \left(\frac{X}{t}\right)^{\frac{1}{2}}\right\}$$

and therefore we find that the same upper bound as for the case $|\alpha| < 1$ holds for $|\alpha| = 1$.

□

Lemma 4.5.6. *Let f as in 4.2 and $|\alpha| \geq 1$. Then, we have for $t \geq 8$*

$$\int_0^{\frac{t}{2}} J_t(y)f(y)\frac{dy}{y} \ll \mathbb{1}_{[2X/3,\infty[}(t)\cdot t^{-\frac{1}{2}}e^{-\frac{2}{5}t},$$

$$\int_{\frac{t}{2}}^{t-t^{\frac{1}{3}}} J_t(y)f(y)\frac{dy}{y} \ll \mathbb{1}_{[\frac{1}{4},\infty[}(X)\mathbb{1}_{[X/3,4X]}(t)\cdot t^{-1}(\log(t))^{\frac{2}{3}},$$

$$\int_{t-t^{\frac{1}{3}}}^{t+t^{\frac{1}{3}}} J_t(y)f(y)\frac{dy}{y} \ll \mathbb{1}_{[\frac{1}{4},\infty[}(X)\mathbb{1}_{[3X/16,3X]}(t)\cdot t^{-1},$$

$$\int_{t+t^{\frac{1}{3}}}^{\infty} J_t(y)f(y)\frac{dy}{y} \ll \mathbb{1}_{[\frac{1}{4},\infty[}(X)\mathbb{1}_{[0,3X/2]}(t)\cdot t^{-1}\min\left\{1+||\alpha|-1|^{-\frac{1}{4}}, \left(\frac{X}{t}\right)^{\frac{1}{2}}\right\},$$

where $\mathbb{1}_{\mathcal{I}}$ is the characteristic function of the interval $\mathcal{I}$.

Proof. We follow the argumentation of the previous lemma. The first three inequalities follow immediately. For the last inequality, we need a lower bound on

$$\min_{\substack{y\geq 1+t^{-\frac{2}{3}} \\ y\asymp X/t}} \left|(|\alpha|-\omega'(y))(y^2-1)^{\frac{1}{4}}y\right| \gg \min_{\substack{y\geq 1+t^{-\frac{2}{3}} \\ y\asymp X/t}} \left(|\alpha|-1+\frac{y-\sqrt{y^2-1}}{y}\right)(y^2-1)^{\frac{1}{4}}y$$

$$\gg \min_{\substack{y\geq 1+t^{-\frac{2}{3}} \\ y\asymp X/t}} \left(|\alpha|-1+\frac{1}{y^2}\right)(y^2-1)^{\frac{1}{4}}y.$$

If $X/t \asymp 1$, then the minimum is at least $|\alpha| t^{-\frac{1}{6}}$, which contributes $t^{-\frac{4}{3}}|\alpha|^{-1} \ll t^{-1}$, otherwise $X/t \gg 1$, in which case the minimum is at least

$$\max\left\{||\alpha|-1|\left(\frac{X}{t}\right)^{\frac{3}{2}}, \left(\frac{X}{t}\right)^{-\frac{1}{2}}\right\} \gg \max\left\{||\alpha|-1|^{\frac{1}{4}}, \left(\frac{X}{t}\right)^{-\frac{1}{2}}\right\},$$

giving a contribution of

$$t^{-\frac{3}{2}}\min\left\{||\alpha|-1|^{-\frac{1}{4}}, \left(\frac{X}{t}\right)^{\frac{1}{2}}\right\}.$$

□

Lemma 4.5.7. *Let f be as in 4.2 and $|\alpha| \leq 1$. Then, we have*

$$\widehat{f}(t) \ll \frac{1+|\log(X)|}{1+X^{\frac{1}{2}}+|1-|\alpha|^2|^{\frac{1}{2}}X}, \qquad \forall t \in \mathbb{R},$$

$$\widehat{f}(t) \ll |t|^{-\frac{3}{2}}\left(1+\min\left\{\left(\frac{X}{|t|}\right)^{\frac{1}{2}}, |1-|\alpha|^2|^{-1}\left(\frac{X}{|t|}\right)^{-\frac{3}{2}}\right\}\right), \qquad \forall |t| \geq 1,$$

$$\widehat{f}(t) \ll \frac{C}{T}|t|^{-\frac{5}{2}}\left(1+\min\left\{\left(\frac{X}{|t|}\right)^{\frac{3}{2}}, |1-|\alpha|^2|^{-2}\left(\frac{X}{|t|}\right)^{-\frac{5}{2}}\right\}\right), \qquad \forall |t| \geq 1.$$

Proof. We follow the proof of [DI83, Lemma 7.1] and [ST09, Prop. 5]. To prove the first inequality we use the equation (A.12)

$$J_{2it}(x) - J_{-2it}(x) = -\frac{4i}{\pi}\sinh \pi t \int_0^\infty \cos(x\cosh\xi)\cos(2t\xi)d\xi.$$

We have by integration by parts

$$\begin{aligned}\int_0^\infty e^{i(\pm x\cosh\xi)}\frac{f(x)}{x}dx &= \int_0^\infty e^{ix(\pm\cosh\xi+\alpha)}\frac{g(x)}{x}dx \\ &= \frac{i}{\pm\cosh\xi+\alpha}\int_0^\infty e^{ix(\pm\cosh\xi+\alpha)}\left(\frac{g(x)}{x}\right)'dx \\ &\ll \min\left\{1, X^{-1}|\cosh\xi \pm \alpha|^{-1}\right\}.\end{aligned}$$

Hence, we find

$$\widehat{f}(t) \ll \int_0^\infty \min\left\{1, X^{-1}|\cosh\xi\pm\alpha|^{-1}\right\}d\xi.$$

Thus, it suffices to bound the latter integral. It is bounded by

$$\begin{aligned}&\ll \int_0^1 \min\left\{1, X^{-1}(\xi^2+1-|\alpha|)^{-1}\right\}d\xi + \int_1^\infty \min\left\{1, X^{-1}e^{-\xi}\right\}d\xi \\ &\ll \int_0^1 \min\left\{1, X^{-1}\xi^{-2}, X^{-1}\xi^{-1}|1-|\alpha||^{-\frac{1}{2}}, X^{-1}|1-|\alpha||^{-1}\right\}d\xi \\ &\quad + \int_1^\infty \min\left\{1, X^{-1}e^{-\xi}\right\}d\xi.\end{aligned}$$

For $X \geq 1$, this is bounded by

$$\ll \min \left\{ X^{-\frac{1}{2}}, \frac{1+\log(X)}{|1-|\alpha||^{\frac{1}{2}} X}, X^{-1}|1-|\alpha||^{-1} \right\} + X^{-1}$$

and for $X \leq 1$, it is bounded by

$$\ll_\epsilon 1 + |\log(X)|.$$

The first inequality follows immediately.

The final two inequalities require some more work. Note that $\widehat{f}(t)$ is even in t. Thus, we can restrict ourselves to $t \geq 1$. We make the substitution $x \to 2tx$ in the definition of $\widehat{f}(t)$

$$\widehat{f}(t) = \frac{i}{\sinh \pi t} \int_0^\infty \frac{J_{2it}(2tx) - J_{-2it}(2tx)}{2} f(2tx) \frac{dx}{x}$$

and use the uniform asymptotic expansion of the function $G_{i\nu}(\nu s)$ from [Dun90, pp. 1009-1010] with $n = 0$:

$$\begin{aligned} G_{2it}(2tx) =& \frac{1}{\sinh(\pi t)} \frac{J_{2it}(2tx) - J_{-2ti}(2tx)}{2i} \\ =& \left(\frac{1}{\pi t}\right)^{\frac{1}{2}} (1+x^2)^{-\frac{1}{4}} \Bigg[\sin(2t\omega(x) - \tfrac{\pi}{4}) \\ & - \cos(2t\omega(x) - \tfrac{\pi}{4}) \frac{3(1+x^2)^{-\frac{1}{2}} - 5(1+x^2)^{-\frac{3}{2}}}{48t} \\ & + \frac{1}{2i} \left(e^{-i\frac{\pi}{4}} \mathcal{E}_{1,1}(2t, \omega(x)) - e^{i\frac{\pi}{4}} \mathcal{E}_{1,2}(2t, \omega(x)) \right) \Bigg], \end{aligned}$$

here

$$\omega(x) = \sqrt{1+x^2} + \log\left(\frac{x}{1+\sqrt{1+x^2}}\right)$$

and the error terms satisfy

$$\mathcal{E}_{1,1}(2t, \omega(x)),\ \mathcal{E}_{1,2}(2t, \omega(x)) \ll |t|^{-\frac{5}{2}} \exp(O(|t|^{-1})).$$

Let us first deal with the error term. The contribution of the error term is bounded by

$$t^{-\frac{5}{2}} \int_0^\infty |f(2tx)| \frac{dx}{x} \ll t^{-\frac{5}{2}} \ll \min\left\{ |t|^{-\frac{3}{2}}, \frac{C}{T} |t|^{-\frac{5}{2}} \right\}.$$

For the remaining summands, we have to deal with integrals of the type

$$t^{-\frac{1}{2}} \int_0^\infty \frac{e^{\pm 2it\omega(x)}}{(1+x^2)^{\frac{1}{4}+\beta}} f(2tx) \frac{dx}{x} = t^{-\frac{1}{2}} \int_0^\infty \frac{e^{2it(\pm\omega(x)+\alpha x)}}{(1+x^2)^{\frac{1}{4}+\beta}} g(2tx) \frac{dx}{x},$$

with $\beta \in \{0, \frac{1}{2}, \frac{3}{2}\}$. We rewrite the above as

$$\frac{1}{2}t^{-\frac{3}{2}} \int_0^\infty \left(e^{2it(\pm\omega(x)+\alpha)} 2t(\pm\omega'(x)+\alpha)\right) \frac{g(2tx)}{x(\pm\omega'(x)+\alpha)(1+x^2)^{\frac{1}{4}+\beta}} dx. \tag{4.37}$$

Since

$$\omega'(x) = \frac{\sqrt{1+x^2}}{x} > 1,$$

we have $\omega'(x) - |\alpha| > 0$. We apply Lemmata 4.5.1 and 4.5.2 with $G(x) = g(2tx)$, $F(x) = e^{2it(\pm\omega(x)+\alpha)} 2t(\pm\omega'(x)+\alpha)$, and $H(x) = [x(\pm\omega'(x)+\alpha)(1+x^2)^{\frac{1}{4}+\beta}]^{-1}$. Moreover, we have

$$\begin{aligned} \min_{x \asymp \frac{X}{t}} \left| x(\pm\omega'(x)+\alpha)(1+x^2)^{\frac{1}{4}+\beta} \right| &\gg \min_{x \asymp \frac{X}{t}} \left| x\left(\frac{1}{x\sqrt{1+x^2}} + 1 - |\alpha|\right)(1+x^2)^{\frac{1}{4}} \right| \\ &\gg \min_{x \asymp \frac{X}{t}} \max\left\{(1+x^2)^{-\frac{1}{4}}, (1-|\alpha|)x(1+x^2)^{\frac{1}{4}}\right\}. \end{aligned}$$

For $x \ll 1$, we see that the function is bounded below by 1. If $x \gg 1$, then the function is bounded by below by

$$\max\left\{\left(\frac{X}{t}\right)^{-\frac{1}{2}}, |1-|\alpha||\left(\frac{X}{t}\right)^{\frac{3}{2}}\right\}$$

Therefore, the integral (4.37) is bounded by

$$t^{-\frac{3}{2}}\left(1 + \min\left\{\left(\frac{X}{t}\right)^{\frac{1}{2}}, |1-|\alpha||^{-1}\left(\frac{X}{t}\right)^{-\frac{3}{2}}\right\}\right).$$

This yields the second inequality. For the third inequality, we proceed from (4.37) with integration by parts. We have to deal with four new integrals

$$\mathcal{I}_1 = t^{-\frac{5}{2}} \int_0^\infty \left(e^{2it(\pm\omega(x)+\alpha)} 2t(\pm\omega'(x)+\alpha)\right) \frac{g(2tx)}{x^2(\pm\omega'(x)+\alpha)^2(1+x^2)^{\frac{1}{4}+\beta}} dx,$$

$$\mathcal{I}_2 = t^{-\frac{5}{2}} \int_0^\infty \left(e^{2it(\pm\omega(x)+\alpha)} 2t(\pm\omega'(x)+\alpha)\right) \frac{g(2tx)(\pm\omega''(x)x^2)}{x^3(\pm\omega'(x)+\alpha)^3(1+x^2)^{\frac{1}{4}+\beta}} dx,$$

$$\mathcal{I}_3 = t^{-\frac{5}{2}} \int_0^\infty \left(e^{2it(\pm\omega(x)+\alpha)} 2t(\pm\omega'(x)+\alpha)\right) \frac{g(2tx)x^2}{x^2(\pm\omega'(x)+\alpha)^2(1+x^2)^{\frac{5}{4}+\beta}} dx,$$

$$\mathcal{I}_4 = t^{-\frac{5}{2}} \int_0^\infty \left(e^{2it(\pm\omega(x)+\alpha)} 2t(\pm\omega'(x)+\alpha)\right) \frac{tx \cdot g'(2tx)}{x^2(\pm\omega'(x)+\alpha)^2(1+x^2)^{\frac{1}{4}+\beta}} dx.$$

Proceeding as before, we find

$$\mathcal{I}_1 \ll t^{-\frac{5}{2}}\left(1+\min\left\{\left(\frac{X}{t}\right)^{\frac{3}{2}}, |1-|\alpha||^{-2}\left(\frac{X}{t}\right)^{-\frac{5}{2}}\right\}\right),$$
$$\mathcal{I}_2 \ll t^{-\frac{5}{2}}\left(1+\min\left\{\left(\frac{X}{t}\right)^{\frac{3}{2}}, |1-|\alpha||^{-3}\left(\frac{X}{t}\right)^{-\frac{9}{2}}\right\}\right),$$
$$\mathcal{I}_3 \ll t^{-\frac{5}{2}}\left(1+\min\left\{\left(\frac{X}{t}\right)^{\frac{3}{2}}, |1-|\alpha||^{-2}\left(\frac{X}{t}\right)^{-\frac{5}{2}}\right\}\right),$$
$$\mathcal{I}_4 \ll \frac{C}{T}t^{-\frac{5}{2}}\left(1+\min\left\{\left(\frac{X}{t}\right)^{\frac{3}{2}}, |1-|\alpha||^{-2}\left(\frac{X}{t}\right)^{-\frac{5}{2}}\right\}\right).$$

We conclude the third inequality from this. □

Lemma 4.5.8. *Let f be as in (4.2) and $|\alpha| \geq 1$. Then, we have*

$$\widehat{f}(t) \ll \frac{1+|\log(X)|+\log(|\alpha|)}{1+X^{\frac{1}{2}}+||\alpha|^2-1|^{\frac{1}{2}}X}, \qquad \forall t \in \mathbb{R}.$$

When $|t| \notin \left[\frac{1}{12}||\alpha|^2-1|^{\frac{1}{2}}X, 2||\alpha|^2-1|^{\frac{1}{2}}X\right]$ and $|t| \geq 1$, we can do better and find in that case

$$\widehat{f}(t) \ll |t|^{-\frac{3}{2}}\left(1+\min\left\{\left(\frac{X}{|t|}\right)^{\frac{1}{2}}, ||\alpha|^2-1|^{-1}\left(\frac{X}{|t|}\right)^{-\frac{3}{2}}\right\}\right),$$
$$\widehat{f}(t) \ll \frac{C}{T}|t|^{-\frac{5}{2}}\left(1+\min\left\{\left(\frac{X}{|t|}\right)^{\frac{3}{2}}, ||\alpha|^2-1|^{-2}\left(\frac{X}{|t|}\right)^{-\frac{5}{2}}\right\}\right).$$

Proof. We follow the proof of the previous lemma which leads us to estimate:

$$\widehat{f}(t) \ll \int_0^\infty \min\left\{1, X^{-1}|\cosh\xi-|\alpha||^{-1}\right\} d\xi.$$

Set $\cosh(\phi) = |\alpha|$ and note that we have $e^\phi \asymp |\alpha|$ and $\log(|\alpha|) \leq \phi \leq 1+\log(|\alpha|)$ for $|\alpha| \geq 1$. This leads to

$$\widehat{f}(t) \ll \int_0^\infty \min\left\{1, X^{-1}\sinh\left(\frac{\xi+\phi}{2}\right)^{-1}\sinh\left(\frac{|\xi-\phi|}{2}\right)^{-1}\right\} d\xi.$$

Hence, it suffices to bound the latter integral. We split up the region of integration into three parts $\mathcal{I}_1, \mathcal{I}_2$ and $\mathcal{I}_3$, where we restrict ourselves to $|\xi-\phi| \geq 1$, $|\xi-\phi| \leq 1 \wedge \xi+\phi \geq 1$ and $|\xi-\phi| \leq 1 \wedge \xi+\phi \leq 1$, respectively. For $X \geq 1$, we have

$$\begin{aligned}\mathcal{I}_1 &\ll \int_0^\infty \min\left\{1, X^{-1}e^{-\max\{\phi,\xi\}}\right\} d\xi \\ &\ll \int_0^\phi \frac{e^{-\phi}}{X}d\xi + \int_\phi^\infty \frac{e^{-\xi}}{X}d\xi \\ &\ll_\epsilon \frac{1+\log(|\alpha|)}{|\alpha|X},\end{aligned}$$

$$\begin{aligned}
\mathcal{I}_2 &\ll \int_{\max\{0,\phi-1\}}^{\phi+1} \min\left\{1, X^{-1}e^{-\frac{\xi+\phi}{2}}|\xi-\phi|^{-1}\right\} d\xi \\
&\ll \int_{-1}^{1} \min\left\{1, X^{-1}e^{-\phi}|\psi|^{-1}\right\} d\psi \\
&\ll \int_0^{\frac{1}{|\alpha|X}} d\psi + \int_{\frac{1}{|\alpha|X}}^{1} \frac{1}{|\alpha|X\psi} d\psi \\
&\ll \frac{1+\log(|\alpha|X)}{|\alpha|X},
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}_3 &\ll \int_{\max\{0,\phi-1\}}^{1-\phi} \min\left\{1, X^{-1}|\xi^2-\phi^2|^{-1}\right\} d\xi \\
&\ll \int_{\max\{-1,-\phi\}}^{1-2\phi} \min\left\{1, X^{-1}\phi^{-1}|\psi|^{-1}, X^{-1}|\psi|^{-2}\right\} d\psi \\
&\ll \mathbb{1}_{[0,1]}(\phi) \min\left\{1, \frac{1+\log^+(X\phi)}{X\phi}, X^{-\frac{1}{2}}\right\} \\
&\ll \mathbb{1}_{[0,1]}(\phi) \min\left\{1, \frac{1+\log(X)}{||\alpha|-1|^{\frac{1}{2}}X}, X^{-\frac{1}{2}}\right\}.
\end{aligned}$$

For $X \leq 1$, we have

$$\begin{aligned}
\mathcal{I}_1 &\ll \int_0^\infty \min\left\{1, X^{-1}e^{-\max\{\phi,\xi\}}\right\} d\xi \\
&\ll \int_0^{\max\{\phi,-\log(X)\}} \min\left\{1, \frac{e^{-\phi}}{X}\right\} d\xi + \int_{\max\{\phi,-\log(X)\}}^\infty \frac{e^{-\xi}}{X} d\xi \\
&\ll_\epsilon \frac{1+\log(|\alpha|)+|\log(X)|}{1+|\alpha|X} + \frac{1}{X}\min\left\{|\alpha|^{-1}, X\right\} \\
&\ll_\epsilon \frac{1+\log(|\alpha|)+|\log(X)|}{1+|\alpha|X},
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}_2 &\ll \int_{\max\{0,\phi-1\}}^{\phi+1} \min\left\{1, X^{-1}e^{-\frac{\xi+\phi}{2}}|\xi-\phi|^{-1}\right\} d\xi \\
&\ll \int_{-1}^{1} \min\left\{1, X^{-1}e^{-\phi}|\psi|^{-1}\right\} d\psi \\
&\ll \min\left\{1, \frac{1+\log^+(|\alpha|X)}{|\alpha|X}\right\},
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}_3 &\ll \int_{\max\{0,\phi-1\}}^{1-\phi} \min\left\{1, X^{-1}|\xi^2-\phi^2|^{-1}\right\} d\xi \\
&\ll \mathbb{1}_{[0,1]}(\phi).
\end{aligned}$$

This completes the case $X \leq 1$.

For the second inequality, we proceed as in Lemma 4.5.7 and have to consider the integral

$$t^{-\frac{1}{2}} \int_0^\infty \frac{e^{2it(\pm\omega(x)+\alpha x)}}{(1+x^2)^{\frac{1}{4}+\beta}} g(2tx) \frac{dx}{x}.$$

We would pick up a stationary phase at $x_0 = (\alpha^2-1)^{-\frac{1}{2}}$. However, we have $x \in [\frac{1}{6}\frac{X}{t}, \frac{X}{t}]$ which does not intersect $[\frac{1}{2}x_0, 2x_0]$. Thus, we split up the integral into two parts $\mathcal{I}_1$ and $\mathcal{I}_2$ corresponding to the intervals $[0, \frac{1}{2}x_0]$ and $[2x_0, \infty[$. Without loss of generality let $\alpha \geq 1$. Assume first that $X/t \leq 1$. In this case, we have by Lemmata 4.5.1 and 4.5.2 with the choice $F(x) = (\pm\omega'(x)+\alpha)e^{2it(\pm\omega(x)+\alpha)}, G(x) = g(2tx)$, and $H(x) = [(1+x^2)^{\frac{1}{4}+\beta}(\sqrt{1+x^2}-\alpha x)]^{-1}$,

$$\mathcal{I}_1 \ll t^{-\frac{3}{2}} \frac{1}{\min\limits_{x\in[0,\frac{1}{2}x_0]\cap[\frac{1}{6}\frac{X}{t},\frac{X}{t}]} \sqrt{1+x^2}-\alpha x},$$

$$\mathcal{I}_2 \ll t^{-\frac{3}{2}} \frac{1}{\min\limits_{x\in[2x_0,\infty[\cap[\frac{1}{6}\frac{X}{t},\frac{X}{t}]} \alpha x-\sqrt{1+x^2}}.$$

The allowed range for t leaves us with two cases, either $x_0 \geq 2\frac{X}{t}$ or $x_0 \leq \frac{1}{12}\frac{X}{t}$. If $x_0 \geq 2\frac{X}{t}$, then the integral over $\mathcal{I}_2$ is 0, and

$$\sqrt{1+x^2}-\alpha x = \frac{1-x^2(\alpha^2-1)}{\sqrt{1+x^2}+\alpha x} \gg 1, \text{ for } x \leq \frac{1}{2}x_0 \text{ and } x \leq 1.$$

Thus, we get a total bound of $t^{-\frac{3}{2}}$. Similarly, if $x_0 \leq \frac{1}{12}\frac{X}{t}$, then we have that the integral over $\mathcal{I}_1$ is 0, and

$$\alpha x-\sqrt{1+x^2} = \frac{x^2(\alpha^2-1)-1}{\sqrt{1+x^2}+\alpha x} \gg \frac{1}{\alpha x}, \text{ for } x \geq 2x_0 \text{ and } x \leq 1.$$

Note that for $x \leq 1$, we also have $\alpha x - \sqrt{1+x^2} \geq \alpha x - \sqrt{2}$ and hence

$$\alpha x-\sqrt{1+x^2} \gg \alpha x + \frac{1}{\alpha x} \text{ for } x \geq 2x_0 \text{ and } x \leq 1.$$

This yields a total bound of $t^{-\frac{3}{2}}$.

Assume now $\frac{X}{t} \geq 1$. In this case, we have

$$\mathcal{I}_1 \ll t^{-\frac{3}{2}} \frac{1}{\min\limits_{x\in[0,\frac{1}{2}x_0]\cap[\frac{1}{6}\frac{X}{t},\frac{X}{t}]} \left(\sqrt{1+x^2}-\alpha x\right)x^{\frac{1}{2}}},$$

$$\mathcal{I}_2 \ll t^{-\frac{3}{2}} \frac{1}{\min\limits_{x\in[2x_0,\infty[\cap[\frac{1}{6}\frac{X}{t},\frac{X}{t}]} \left(\alpha x-\sqrt{1+x^2}\right)x^{\frac{1}{2}}}.$$

If $x_0 \geq 2\frac{X}{t}$, then we have that the integral over $\mathcal{I}_2$ is 0, and

$$\left(\sqrt{1+x^2}-\alpha x\right)x^{\frac{1}{2}} = \frac{1-x^2(\alpha^2-1)}{\sqrt{1+x^2}+\alpha x}x^{\frac{1}{2}} \gg x^{-\frac{1}{2}}, \text{ for } x \leq \frac{1}{2}x_0 \text{ and } x \geq \frac{1}{12}.$$

Thus, we get a total bound of $t^{-\frac{3}{2}}\left(\frac{X}{t}\right)^{\frac{1}{2}} \ll t^{-\frac{3}{2}}|\alpha^2-1|^{-1}\left(\frac{X}{t}\right)^{-\frac{3}{2}}$. Similarly, if $x_0 \leq \frac{1}{12}\frac{X}{t}$ we have that the integral over $\mathcal{I}_1$ is 0, and

$$\left(\alpha x-\sqrt{1+x^2}\right) = \frac{x^2(\alpha^2-1)-1}{\sqrt{1+x^2}+\alpha x} \geq \frac{3}{8}\frac{x^2(\alpha^2-1)}{\alpha x} \gg \frac{1}{\alpha x}, \text{ for } x \geq 2x_0 \text{ and } x \geq \frac{1}{6}.$$

This yields a total bound of

$$t^{-\frac{3}{2}} \cdot \min\left\{\alpha\left(\frac{X}{t}\right)^{\frac{1}{2}}, \frac{\alpha}{\alpha^2-1}\left(\frac{X}{t}\right)^{-\frac{3}{2}}\right\} \ll t^{-\frac{3}{2}}\left(1+\min\left\{\left(\frac{X}{t}\right)^{\frac{1}{2}}, \frac{1}{\alpha^2-1}\left(\frac{X}{t}\right)^{-\frac{3}{2}}\right\}\right),$$

since $\frac{X}{t} \geq 1$. This proves the second inequality.

For the third inequality, we integrate once by parts. This leads us to consider the integrals

$$\begin{aligned}
\mathcal{I}_4 &= t^{-\frac{5}{2}} \int_0^\infty \left(e^{2it(\pm\omega(x)+\alpha)} 2t(\pm\omega'(x)+\alpha)\right) \frac{g(2tx)}{x^2(\pm\omega'(x)+\alpha)^2(1+x^2)^{\frac{1}{4}+\beta}} dx, \\
\mathcal{I}_5 &= t^{-\frac{5}{2}} \int_0^\infty \left(e^{2it(\pm\omega(x)+\alpha)} 2t(\pm\omega'(x)+\alpha)\right) \frac{g(2tx)(\pm\omega''(x)x^2)}{x^3(\pm\omega'(x)+\alpha)^3(1+x^2)^{\frac{1}{4}+\beta}} dx, \\
\mathcal{I}_6 &= t^{-\frac{5}{2}} \int_0^\infty \left(e^{2it(\pm\omega(x)+\alpha)} 2t(\pm\omega'(x)+\alpha)\right) \frac{g(2tx)x^2}{x^2(\pm\omega'(x)+\alpha)^2(1+x^2)^{\frac{5}{4}+\beta}} dx, \\
\mathcal{I}_7 &= t^{-\frac{5}{2}} \int_0^\infty \left(e^{2it(\pm\omega(x)+\alpha)} 2t(\pm\omega'(x)+\alpha)\right) \frac{tx \cdot g'(2tx)}{x^2(\pm\omega'(x)+\alpha)^2(1+x^2)^{\frac{1}{4}+\beta}} dx.
\end{aligned}$$

By similar means as before, we have that

$$\begin{aligned}
\mathcal{I}_4 &\ll t^{-\frac{5}{2}}\left(1+\min\left\{\left(\frac{X}{t}\right)^{\frac{3}{2}}, ||\alpha|^2-1|^{-2}\left(\frac{X}{t}\right)^{-\frac{5}{2}}\right\}\right), \\
\mathcal{I}_5 &\ll t^{-\frac{5}{2}}\left(1+\min\left\{\left(\frac{X}{t}\right)^{\frac{3}{2}}, ||\alpha|^2-1|^{-3}\left(\frac{X}{t}\right)^{-\frac{9}{2}}\right\}\right), \\
\mathcal{I}_6 &\ll t^{-\frac{5}{2}}\left(1+\min\left\{\left(\frac{X}{t}\right)^{\frac{3}{2}}, ||\alpha|^2-1|^{-2}\left(\frac{X}{t}\right)^{-\frac{5}{2}}\right\}\right), \\
\mathcal{I}_7 &\ll \frac{C}{T} t^{-\frac{5}{2}}\left(1+\min\left\{\left(\frac{X}{t}\right)^{\frac{3}{2}}, ||\alpha|^2-1|^{-2}\left(\frac{X}{t}\right)^{-\frac{5}{2}}\right\}\right).
\end{aligned}$$

We conclude the last inequality from this.

□

Lemma 4.5.9. *Let f be as in (4.2). For $0 \leq t \leq \frac{1}{4} - \delta$, we have the following expansion*

$$\widehat{f}(it) = -\frac{1}{2}\int_{\frac{X}{2}}^{X} Y_{2t}(x) e^{i\alpha x} \frac{dx}{x} + O_{\epsilon,\delta}\left(1 + \frac{T}{C} X^{-2t-\epsilon}\right).$$

Proof. We have

$$\begin{aligned}
\widehat{f}(it) &= \frac{1}{\sin(2\pi t)} \int_0^\infty \frac{J_{-2t}(x) - J_{2t}(x)}{2} f(x) \frac{dx}{x} \\
&= -\frac{1}{2}\int_0^\infty \left[\frac{J_{2t}(x)\cos(2\pi t) - J_{-2t}(x)}{\sin(2\pi t)} + \frac{J_{2t}(x) - J_{2t}(x)\cos(2\pi t)}{\sin(2\pi t)}\right] f(x) \frac{dx}{x} \\
&= -\frac{1}{2}\int_0^\infty \left[Y_{2t}(x) + J_{2t}(x)\tan(\pi t)\right] f(x) \frac{dx}{x}.
\end{aligned}$$

Now, we have

$$\int_0^\infty J_{2t}(x)\tan(\pi t)f(x)\frac{dx}{x} \ll \int_0^\infty \min\left\{x^{2t}, x^{-\frac{1}{2}}\right\}\frac{g(x)}{x}dx \ll 1$$

and

$$\left(\int_{\frac{2\pi\sqrt{mn}}{(C+T)}}^{\frac{X}{2}} + \int_X^{\frac{4\pi\sqrt{mn}}{(C-T)}}\right) Y_{2t}(x)f(x)\frac{dx}{x} \ll \frac{T}{C}\sup_{x\sim X}|Y_{2t}(x)|.$$

The following inequality will imply the result

$$|Y_{2t}(x)| \ll_\epsilon \begin{cases} x^{-2t-\epsilon}, & \text{if } x \le 1, \\ x^{-\frac{1}{2}}, & \text{if } x \ge 1. \end{cases}$$

The range $x \ge 1$ follows from (A.19) and for the range $x \le 1$, we make use of the following integral representation (A.13):

$$Y_{2t}(x) = -\frac{2(\frac{x}{2})^{-2t}}{\sqrt{\pi}\Gamma(\frac{1}{2}-2t)}\int_1^\infty \frac{\cos(xy)}{(y^2-1)^{2t+\frac{1}{2}}}dy.$$

The integral from 1 to $\frac{1}{x}$ is bounded by

$$\int_1^2 \frac{1}{(y-1)^{1-2\delta}}dy + \int_2^{\max\{2,\frac{1}{x}\}} \frac{1}{(y^2-1)^{\frac{1}{2}}}dy$$
$$= \frac{1}{2\delta}(y-1)^{2\delta}\Big|_{y=1}^{2} + \log\left(\sqrt{y^2-1}+y\right)\Big|_{y=2}^{\max\{2,\frac{1}{x}\}} \ll_{\epsilon,\delta} x^{-\epsilon}$$

and the remaining integral is bounded by $O(1)$ by Lemma 4.5.1 with $F(y) = \cos(xy)$ and $G(y) = (y^2-1)^{2t+\frac{1}{2}}$. □

5

THE CIRCLE METHOD

5.1 INTRODUCTION

As already mentioned in the introduction of this thesis, the Hardy–Littlewood circle method finds its origin in a paper by Hardy and Ramanujan [HR18] on the asymptotic behaviour of the size of partition numbers. The idea that led them to success was the following. Suppose we wish to understand the elements of a sequence $(a_n)_{n\in\mathbb{N}_0}$ and we understand to some degree its generating series

$$F(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Say, it has radius of convergence $R > 0$. Then, we may write the element a_n as the contour integral

$$a_n = \int_{|z|=r} F(z) z^{-n-1} dz,$$

for any $0 < r < R$. The Hardy–Littlewood philosophy now dictates that $F(z)$ is generically small as $|z| \to R$, unless z is close to $R \cdot e(\frac{a}{q})$ with $\frac{a}{q} \in \mathbb{Q}$, in which case we may hope for an asymptotic expansion around the point $R \cdot e(\frac{a}{q})$. The former is known as the minor arcs and the latter as major arcs. If all of the above is satisfied in a sufficiently quantitative manner, then one may derive an asymptotic expansion for a_n with a dominant contribution coming from the behaviour of $F(z)$ near $R \cdot e(\frac{a}{q})$ with q not too large.

Contributions from Vinogradov [Vin28] simplified the matter significantly by showing that often one does not require the full generating series, but a partial sum suffices. By doing so, all issues that came from taking the limit $r \to R$ suddenly disappeared. Another large contribution by Vinogradov [Vin35b, Vin35a] was the introduction of his mean value. It is known as Vinogradov's mean value and it constitutes a powerful tool to deal with exponential sums. In particular, it used to prove the best known zero-free region of the Riemann zeta function - more on this in Section 5.3.

Nowadays, the circle method takes on various shapes and has been applied in all sorts of situations. Kloosterman [Klo26] developed a minor-arc-free circle method based on a Farey dissection of the unit interval. Rademacher [Rad43] used Ford circles to extend the Hardy–Ramanujan argument for the asymptotic behaviour of the partition function to a complete expansion as a series. Jutila [Jut99] developed a circle method with overlapping intervals and much freedom, which came at the cost of having an L^2-error-term. Duke–Friedlander–Iwaniec [DFI93] developed the smooth delta symbol circle method and applied it to automorphic forms. Heath-Brown [HB96] then further applied it to Diophantine equations. Recently, Munshi [Mun15] had success with an automorphic representation of the delta symbol (see Theorem 3.7.6).

5.2 SMOOTH DELTA SYMBOL CIRCLE METHOD

The smooth delta symbol circle method is essentially due to Duke–Friedlander–Iwaniec [DFI93], but is based on Kloosterman's version of the circle method [Klo26]. The version we present here was worked out by Heath-Brown [HB96] in order find integer solutions to Diophantine equations.

We start with a positive smooth bump function ω_0 that is supported on $[\frac{1}{2}, 1]$ and satisfies

$$\int_{-\infty}^{\infty} \omega_0(x)dx = 1.$$

Let $Q > 1$ be any real number and observe that if $n \in \mathbb{Z}$ is a non-zero integer, then $q \mapsto \frac{n}{q}$ is an involution on the set of divisors of n. Hence, we find

$$\sum_{q|n} \left(\omega_0 \left(\frac{q}{Q} \right) - \omega_0 \left(\frac{n}{qQ} \right) \right) = 0.$$

However, if $n = 0$ we find

$$\sum_{q|0} \left(\omega_0 \left(\frac{q}{Q} \right) - \omega_0(0) \right) = \sum_{q \in \mathbb{Z}} \omega_0 \left(\frac{q}{Q} \right) = \sum_{q \in \mathbb{Z}} \int_{-\infty}^{\infty} \omega_0 \left(\frac{x}{Q} \right) e(-qx)dx, \tag{5.1}$$

by Poisson summation. The latter has a dominant term of Q for $q = 0$ and the remaining terms are $O_A(Q(|q|Q)^{-A})$ for any $A \geq 0$, which follows from integration by parts.

Therefore, we may write (5.1) as $c_Q^{-1}Q$ with $c_Q = 1 + O_A(Q^{-A})$. In conclusion, we have shown that

$$\frac{c_Q}{Q}\sum_{q|n}\left(\omega_0\left(\frac{q}{Q}\right) - \omega_0\left(\frac{n}{qQ}\right)\right) = \delta(n) = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0. \end{cases} \tag{5.2}$$

We may restrict ourselves to positive divisors q if we replace n by $|n|$ in $\omega_0(\frac{n}{qQ})$. By further resolving the divisibility $q|n$ by means of additive characters, we find that (5.2) is equal to

$$\frac{c_Q}{Q}\sum_{q=1}^{\infty}\frac{1}{q}\sum_{a \bmod (q)} e(\tfrac{an}{q})\left(\omega_0\left(\frac{q}{Q}\right) - \omega_0\left(\frac{n}{qQ}\right)\right) = \frac{c_Q}{Q^2}\sum_{q=1}^{\infty}\sideset{}{'}\sum_{a \bmod (q)} e(\tfrac{an}{q})h\left(\frac{q}{Q}, \frac{n}{Q^2}\right), \tag{5.3}$$

where

$$h(x,y) = \sum_{j=1}^{\infty}\frac{1}{xj}\left(\omega_0(xj) - \omega_0\left(\frac{|y|}{xj}\right)\right). \tag{5.4}$$

One can now apply this to a weighted counting of solutions to a Diophantine equation $F(\boldsymbol{x}) = 0$:

$$N(F,\omega) = \sum_{\substack{\boldsymbol{x}\in\mathbb{Z}^n \\ F(\boldsymbol{x})=0}} \omega(\boldsymbol{x}),$$

where ω is a compactly supported bounded function.

Theorem 5.2.1. *We have*

$$N(F,\omega) = \frac{c_Q}{Q^2}\sum_{q=1}^{\infty}\sideset{}{'}\sum_{a \bmod (q)}\sum_{\boldsymbol{x}\in\mathbb{Z}^n}\omega(\boldsymbol{x})e\left(\tfrac{aF(\boldsymbol{x})}{q}\right)h\left(\frac{q}{Q}, \frac{F(\boldsymbol{x})}{Q^2}\right),$$

for any $Q > 1$. Moreover, if ω is smooth and compactly supported, then

$$N(F,\omega) = \frac{c_Q}{Q^2}\sum_{\boldsymbol{c}\in\mathbb{Z}^n}\sum_{q=1}^{\infty}\frac{1}{q^n}S_q(\boldsymbol{c})I_q(\boldsymbol{c}),$$

where

$$S_q(\boldsymbol{c}) = \sideset{}{'}\sum_{a \bmod (q)}\sum_{\boldsymbol{b}\in\mathbb{Z}^n/q\mathbb{Z}^n} e\left(\frac{aF(\boldsymbol{b}) + \boldsymbol{c}\cdot\boldsymbol{b}}{q}\right),$$

$$I_q(\boldsymbol{c}) = \int_{\mathbb{R}^n}\omega(\boldsymbol{x})h\left(\frac{q}{Q}, \frac{F(\boldsymbol{x})}{Q^2}\right)e(-\tfrac{\boldsymbol{c}\cdot\boldsymbol{x}}{q})d\boldsymbol{x}.$$

Proof. This is [HB96, Thm 2] or it simply follows from the previous discussion and Poisson summation applied to all arithmetic progressions $\boldsymbol{x} \equiv \boldsymbol{b} \bmod (q)$. □

The function h approximates a delta function in the following sense.

Lemma 5.2.2. *Let $f \in L^1(\mathbb{R})$ be a smooth function. Then, for $x \ll 1$ and $M \in \mathbb{N}_0$, we have*

$$\int_{-\infty}^{\infty} f(y)h(x,y)dy = f(0) + O_M\left(x^{M-1}\|f\|_{M,1}\right),$$

where $\|f\|_{M,1}$ denotes the Sobolev norm on $L^1(\mathbb{R})$ of order M.

Proof. This is [IK04, Cor. 20.19]. □

The following estimates on the function h come in handy as well.

Lemma 5.2.3. *The function $h(x,y)$ vanishes, unless $x \leq \max\{1, 2|y|\}$, in which case we have*

$$\frac{d^{m+n}}{dx^m dy^n} h(x,y) \ll_{N,m,n} x^{-1-m-n}\left(x^N + \min\left\{1, \left(\frac{x}{|y|}\right)^N\right\}\right),$$

for $m, n, N \in \mathbb{N}_0$. The term x^N on the right-hand side may be omitted for $n \neq 0$.

Proof. This is [HB96, Lem. 4,5] □

5.3 EFFECTIVE VINOGRADOV MEAN VALUE THEOREM

This section is directly taken from our previous work [Ste16].

Let k and s denote two natural numbers. Vinogradov's mean value theorem is a bound on the integer solutions of the Diophantine equation

$$\sum_{i=1}^{s} x_i^j = \sum_{i=1}^{s} y_i^j, \quad (j = 1, \dots, k), \tag{5.5}$$

with $0 < \boldsymbol{x}, \boldsymbol{y} \leq X$. By orthogonality, the number of solutions is equal to

$$J_{s,k}(X) = \int_{[0,1[^k} |f(X/2, X, \boldsymbol{\alpha})|^{2s} d\boldsymbol{\alpha},$$

where we define

$$f(N, M, \boldsymbol{\alpha}) = \sum_{N-\frac{1}{2}M < x \leq N+\frac{1}{2}M} e(\boldsymbol{\alpha} \cdot \boldsymbol{\vartheta}(x))$$

for real N, M with $M \geq 1$ and $\boldsymbol{\vartheta}(x) = (x, x^2, \dots, x^k)$. Lower bounds for $J_{s,k}(X)$ are well known and easily proved (see for example [Vau97]). They admit the form

$$J_{s,k}(X) \gg_{s,k} \max\{X^s, X^{2s-\frac{1}{2}k(k+1)}\}. \tag{5.6}$$

In a recent breakthrough, Bourgain, Demeter, and Guth [BDG15] have shown that (5.6) is sharp up to a factor X^ϵ; i.e. they have proven the inequality

$$J_{s,k}(X) \ll_{s,k,\epsilon} \max\{X^{s+\epsilon}, X^{2s-\frac{1}{2}k(k+1)+\epsilon}\} \tag{5.7}$$

to hold for all $s, k \in \mathbb{N}$ and $\epsilon > 0$. An application of the circle method further shows that one has an asymptotic of the shape

$$J_{s,k}(X) \sim C_{s,k} X^{2s - \frac{1}{2}k(k+1)} \tag{5.8}$$

for all $s > \frac{1}{2}k(k+1)$. Before this latest breakthrough, there has been a long history of improvements towards (5.7) and (5.8). Following Vinogradov [Vin35a], who gave an estimate of the shape

$$J_{s,k}(X) \ll_{s,k} X^{2s - \frac{1}{2}k(k+1) + \eta_{s,k}},$$

there have been improvements in the argument by Linnik [Lin43], Karatsuba [Kar73] and Stechkin [Ste75] leading to an error in the exponent of only $\eta_{s,k} = \frac{1}{2}k^2(1 - 1/k)^{\lfloor s/k \rfloor}$. This allows one to get the asymptotic (5.8) as soon as $s \geq 3k^2(\log k + O(\log\log k))$. Wooley [Woo92] was further able to decrease the exponent to roughly $\eta_{s,k} = k^2 e^{-2s/k^2}$ by using his efficient differencing method, an extension of Linnik's argument, which allowed him to show that the asymptotic (5.8) holds for $s \geq k^2(\log k + O(\log\log k))$. Later, Wooley [Woo12] developed a powerful new argument, called efficient congruencing, which enabled him to prove (5.7) for $s \geq k(k+1)$. Note that this is just a factor of 2 off the critical case $s = \frac{1}{2}k(k+1)$, from which all other cases would follow. There has followed a series of papers in which Wooley has refined his method, leading to proofs of (5.7) for $s \leq \frac{1}{2}k(k+1) - \frac{1}{3}k + O(k^{\frac{2}{3}})$ [Woo17a] and a full proof when $k = 3$ [Woo16]. The history of the main conjecture (5.7) ends with Bourgain, Demeter, and Guth's full proof using decoupling theory from harmonic analysis.

Vinogradov's mean value theorem has a broad range of applications. For example, it can be used to get strong bounds on exponential sums (see Chapter 8.5 in [IK04]). These strong bounds can then be used to get zero-free regions of the Riemann zeta function, something which has been made explicit by Ford [For02]. Furthermore, they have been used implicitly by Halász and Turán [HT69] to get zero-density estimates for the Riemann zeta function. Other applications of Vinogradov's mean value theorem include estimates for short mixed character sums, such as found in work of Heath-Brown and Pierce [HBP15] and Kerr [Ker14], as well as contributions to restriction theory worked out by Wooley [Woo17b] and Bourgain, Demeter and Guth [BDG15]. In all of these applications it is desirable to have an effective version of Vinogradov's mean value theorem.

Effective versions have been given by Hua [Hua49], whose argument is based on Vinogradov's original method, Stechkin [Ste75] as well as by Arkhipov, Chubarikov and Karatsuba [ACK04], whose work is based on Linnik's p-adic argument, and Ford [For02], whose argument is based on Wooley's efficient differencing method. In this section, we prove effective bounds using Wooley's efficient congruencing method combined with the older arguments of Vinogradov and Hua.

Let us give an overview of the heart of Vinogradov's and Hua's argument. If we have two tuples $\boldsymbol{x}, \boldsymbol{x'}$ such that $\|\boldsymbol{x} - \boldsymbol{x'}\|_\infty \leq S$, then we have

$$\left| \sum_{i=1}^{k} (x_i^j - x_i'^j) \right| \leq jk \cdot SX^{j-1}. \tag{5.9}$$

The question of whether this can be reversed arises naturally and the answer is in the affirmative, although it depends on how well-spaced $\boldsymbol{x}$ is; i.e. how large $\min_{i \neq j} |x_i - x_j|$ is (see Lemma 5.3.5). In his paper, Hua [Hua49] uses this reversibility by writing (5.5) as

$$\sum_{i=1}^{k} (x_i^j - x_i'^j) = \sum_{i=1}^{s-k} (y_i^j - y_i'^j). \tag{5.10}$$

By splitting up the $\boldsymbol{y}$ and $\boldsymbol{y'}$ into $X^{\frac{2(s-k)}{k}}$ intervals of length at most $X^{1-\frac{1}{k}}$ and using the integer translation invariance in combination with Hölder's inequality, one can force $\|\boldsymbol{y}\|_\infty, \|\boldsymbol{y'}\|_\infty \leq X^{1-\frac{1}{k}}$ in (5.10). Now, the right-hand side of (5.10) is small. It is in fact at most $(s-k)X^{j-1}X^{1-\frac{j}{k}}$. By splitting up the right-hand side further into $(s-k)X^{1-\frac{j}{k}}$ intervals of size X^{j-1} and using Cauchy–Schwarz, one is able to reduce to (5.9) with $S = 1$. Fixing the $\boldsymbol{x'}$ arbitrarily allows now only $O_{s,k}(1)$ choices for the $\boldsymbol{x}$ as $x_i = x_i' + O_{s,k}(1)$ and the choices for $\boldsymbol{y}$ and $\boldsymbol{y'}$ can be bounded by $J_{s-k,k}(X^{1-\frac{1}{k}})$. This gives

$$J_{s,k}(X) \ll_{s,k} \log(2X)^2 \cdot X^{\frac{2(s-k)}{k}} \cdot \prod_{j=1}^{k} X^{1-\frac{j}{k}} \cdot X^k \cdot J_{s-k,k}(X^{1-\frac{1}{k}}),$$

where the $\log(2X)^2$ is coming from a dyadic argument ensuring that $\min_{i \neq j} |x_i - x_j|$ is not too small. By iterating this inequality l-times, Hua proved the following upper bound for $s \geq \frac{1}{4}k(k+1) + lk$:

$$J_{s,k}(X) \leq (7s)^{4sl} \log(X)^{2l} X^{2s - \frac{1}{2}k(k+1) + \frac{1}{2}k(k+1)(1-\frac{1}{k})^l} \quad \forall X \geq 2.$$

The same kind of argument also works if we only force $\|\boldsymbol{y}\|_\infty, \|\boldsymbol{y'}\|_\infty \leq X^{1-\theta}$, with θ tiny. In this case, one concludes $x_i = x_i' + O_{s,k}(X^{1-k\theta})$ and one can put the $\boldsymbol{x}$'s into a box of size $X^{1-k\theta}$. Now, Wooley's efficient machinery lets us interchange the roles of $\boldsymbol{x}$ and $\boldsymbol{y}$ and thus allows us to play the same game again with $k\theta$ instead of θ. In every iteration, there is a slight gain in the exponent, depending on s, k, θ and $\eta_{s,k}$. In the simplest form of Wooley's efficient machinery, these gains stack up to overcome the defect of the method as soon as $s \geq k(k+1)$, leading to a slight decrease of $\eta_{s,k}$ as seen in the following theorem.

Theorem 5.3.1. *Let $s, k \in \mathbb{N}$ with $k \geq 3$ and $2\log(k) \geq \lambda = \frac{s-k}{k^2} \geq 1$. Assume that*

$$J_{s,k}(X) \leq C \log_2(2X)^{\delta} X^{2s-\frac{1}{2}k(k+1)+\eta} \quad \forall X \geq 1,$$

for some $0 \leq \delta$ and $0 < \eta \leq \frac{1}{2}k(k+1)$. Further, let

$$D \geq \max\left\{1, \frac{\log\left(\frac{k^2}{2\eta}\frac{\lambda-1}{\lambda^2}+1\right)}{\log(\lambda)}\right\}$$

be an integer and set $\theta = k^{-(D+1)}$. Then, we have

$$\begin{aligned} J_{s,k}(X) \leq & C \cdot 2^{\frac{3}{2}k^2+\frac{11}{2}k+1} k^{\frac{1}{2}k^2+\frac{25}{6}k-2} \cdot \mathcal{M}_0 \\ & \cdot \log_2(2X)^{\delta+\frac{2\lambda k-1}{\lambda k-1}} X^{2s-\frac{1}{2}k(k+1)+\eta} \cdot X^{-\eta\theta\frac{s-2k}{s-k}}, \qquad \forall X \geq 1, \end{aligned}$$

where $\mathcal{M}_0$ is defined as follows

$$\begin{aligned} \mathcal{M}_0 &= \max_{\gamma \in \{1, \frac{s-k}{s-2k}\}} \left\{ \left(2^{\frac{1}{2}k^2+\frac{31}{6}k+7} e^{\frac{3}{4}k^2-\frac{1}{2}k} k^{-\frac{1}{2}k^2+\frac{25}{3}k}\right)^{\gamma}, 2^{-\frac{1}{2}k^2-\frac{11}{6}k} \right\} \\ &= \begin{cases} \left(2^{\frac{1}{2}k^2+\frac{31}{6}k+7} e^{\frac{3}{4}k^2-\frac{1}{2}k} k^{-\frac{1}{2}k^2+\frac{25}{3}k}\right)^{\frac{s-k}{s-2k}}, & k \leq 43, \\ 2^{\frac{1}{2}k^2+\frac{31}{6}k+7} e^{\frac{3}{4}k^2-\frac{1}{2}k} k^{-\frac{1}{2}k^2+\frac{25}{3}k}, & 44 \leq k \leq 62, \\ 2^{-\frac{1}{2}k^2-\frac{11}{6}k}, & k \geq 63. \end{cases} \end{aligned} \tag{5.11}$$

By iterating this theorem and combining it with the Hardy–Littlewood method, one may conclude the following result, which is a special case of Theorem 5.3.22.

Theorem 5.3.2. *Let $k \geq 3$, $s \geq \frac{5}{2}k^2 + k$. Furthermore, let $X \geq s^{10}$. Then, we have the estimate*

$$J_{s,k}(X) \leq C X^{2s-\frac{1}{2}k(k+1)},$$

where C is the maximum of $4k^{30k^3}$ and

$$\left[2^{\frac{3}{2}k^2+\frac{11}{2}k+1+D} k^{\frac{1}{2}k^2+\frac{25}{6}k-2+D} \mathcal{M}_0\right]^{\frac{33}{10}k^{D+1}} \cdot 4(2k)^{2k^3+11k^2},$$

where

$$D = \left\lceil \frac{2\log(k) + \log(\log(k)) + 4.2}{\log(2)} \right\rceil$$

and $\mathcal{M}_0$ *as in* (5.11).

Although Wooley's method is in principle effective and can be made effective in a similar fashion as we do here, it has a rather big disadvantage. Namely, the conditioning and the congruencing step get into each others way. This may be seen best in [Woo12, Section 7], where the sequence $\{b_n\}_n$ follows the iteration scheme $b_{n+1} = kb_n + h_n$ and a lot of effort is put into showing that this sequence doesn't grow too fast. A consequence of this is that the parameter θ has to be smaller by a factor of 2, which may be further improved down to $\frac{4}{3}$. However, this decrease in θ affects the speed of convergence of $\eta_{s,k}$ drastically (see Theorem 5.3.1). By using the techniques of Vinogradov and Hua instead, gains us an independence of the conditioning/well-spacing and the congruencing/boxing step, which leaves us with a simple iteration scheme $b_{n+1} = kb_n$. It is this simple iteration scheme which makes the rather basic outline of the proof in Section 5.3.2 clean. Clean in the sense that simply specifying the involved parameters as well as analysing the dependence in $\boldsymbol{g}$, which simply boils down to an exponent being non-positive, would yield a complete proof. This is not the case when working with congruences. Another novelty of the simple iteration scheme is that it allows the introduction of the parameter $\lambda = \frac{s-k}{k^2}$. This parameter has a large impact on the number of iterations needed in order to decrease the exponent, which has a welcoming effect on the constant. From Theorem 5.3.1, it can easily be seen that choosing $\lambda > 1$ rather than $\lambda = 1$ decreases the number of iterations D from polynomially in k down to logarithmically in k. This has the effect of reducing the constant from $k^{k^{O(k^2/\epsilon)}}$ down to $k^{k^{O(\log(k^2/\epsilon)/\log(\lambda))}}$ if one wishes to achieve an exponent $\eta_{s,k} \leq \epsilon$. This explains why we chose to present Theorem 5.3.2 with a slightly larger s than the method would allow.

In view of further improvements in efficient congruencing and the recent breakthrough by Bourgain, Demeter, and Guth, one might ask to which extent they can be made effective and how such a result would compare to Theorem 5.3.2. Let us first remark that the proof of Bourgain, Demeter, and Guth follows a similar iteration scheme as multigrade efficient congruencing. One iteration of theirs shows that the quantity $V_{p,n}(\delta)$ in their paper, which may be compared to X^η in our setting, can be replaced by

$$\delta^{-\frac{u}{2}} V_{p,n}(\delta)^{1-uW},$$

whereas efficient congruencing/boxing shows that X^{η} can be replaced by

$$X^{\eta}(X^{\Delta\theta} + X^{-\eta\theta\frac{s-2k}{s-k}})$$

(see Proposition 5.3.19). Now, their parameter u can be compared with the parameter θ in efficient congruencing in the sense that they have to look at the tiniest scales as well in order for their iteration scheme to go through. Furthermore, the factor $\delta^{-\frac{u}{2}}$ can be seen as the defect of the method, which one also gets in efficient congruencing (compare to the positive term in Δ). Now, the tinier $V_{p,n}(\delta)$ gets the larger W has to be and henceforth more iterations are needed to further decrease $V_{p,n}(\delta)$. This is the same kind of problem that also arises in effective versions of efficient congruencing/boxing.

Now, we should remark that in most applications of Vinogradov's mean value theorem it is important that s is above the critical case; i.e. $s \geq \frac{1}{2}k(k+1)$. Later versions of efficient congruencing as well as the proof of the main conjecture by Bourgain, Demeter, and Guth attack the problem from below, which corresponds to the case $\lambda = 1$. Therefore, constants of the size $k^{k^{O(k^2/\epsilon)}}$ should be expected. But, let us suppose now that Bourgain, Demeter, and Guth's proof could be adapted to an attack from above. This would then lead to a doubling of the parameter λ, which would speed up the rate of convergence significantly and thus decrease the implied constant.

It is natural to ask if and to what extent Theorem 5.3.2 leads to improvements of the explicit zero-free region of the Riemann zeta function, as in Ford's work [For02]. Unfortunately, the answer is that there are no direct improvements. The reason for this is the growth of the constant. The dominating term is roughly $k^{k^{O(\log(k)/\log(\lambda))}}$, which is a lot bigger compared to the term $k^{O(k^3)}$ appearing in Ford's work. When it comes to its application, one only takes the k^4-th root and thus the constant is too large. There may however be a way around this. Similar to the argument in [Woo12, Section 2], one may choose $D = 1$ and replace Proposition 5.3.16 with one that bounds the quantity at hand in terms of $J_{s-k}(X^{1-\frac{1}{k}})$. This leads to an error of the exponent morally of the size $\eta_{s,k} = k^2 e^{-\frac{1}{2}(s/k^2)^2}$, whilst keeping the constant on the scale $k^{O(k^4)}$. Then, taking the s^2-th root, with $s = k^2 \log(k)^{\frac{1}{2}}$ rather than $s = k^2$, leads to a zero-free region, which is asymptotically slightly worse than Ford's explicit zero-free region. It is therefore not clear if such an endeavour would be fruitful and lead to an improved zero-free region of the Riemann zeta function in an intermediate range.

5.3.1 *Notation*

As already introduced, we let

$$f(N, M, \boldsymbol{\alpha}) = \sum_{N-\frac{1}{2}M<x\leq N+\frac{1}{2}M} e(\boldsymbol{\alpha} \cdot \boldsymbol{\vartheta}(x))$$

for real N, M with $M \geq 1$, where $\boldsymbol{\vartheta}(x) = (x, x^2, \ldots, x^k)$. Furthermore, we call an interval of the shape $]\boldsymbol{a}, \boldsymbol{b}] =]a_1, b_1] \times \cdots \times]a_n, b_n]$ a box. By $\mathfrak{B}^n(\boldsymbol{N}, \boldsymbol{M})$, we denote the box

$$\mathfrak{B}^n(\boldsymbol{N}, \boldsymbol{M}) = \prod_{i=1}^{n} \left] N_i - \frac{1}{2}M_i, N_i + \frac{1}{2}M_i \right] .$$

Furthermore, we allow ourselves to abuse some notation here: any numbers, say N, M in the argument of $\mathfrak{B}^n(N, M)$ are to be regarded as n-dimensional vectors with entry N, respectively M, in each coordinate. To a box $\mathfrak{B}^n(\boldsymbol{N}, \boldsymbol{M})$, we associate the product

$$\mathfrak{F}^n(\boldsymbol{N}, \boldsymbol{M}, \boldsymbol{\alpha}) = \prod_{i=1}^{n} f(N_i, M_i, \boldsymbol{\alpha}). \tag{5.12}$$

We say a box $\mathfrak{B}^n(\boldsymbol{N}, \boldsymbol{M})$ is R-well-spaced if $|N_i - N_j| \geq 2R$ for all $i \neq j$ and $1 \leq M_i \leq R$ for $i = 1, \ldots, n$. In this case, we adjust the definition (5.12) to

$$\mathfrak{F}^n_R(\boldsymbol{N}, \boldsymbol{M}, \boldsymbol{\alpha}) = \prod_{i=1}^{n} f(N_i, M_i, \boldsymbol{\alpha})$$

to further indicate that the box $\mathfrak{B}^n(\boldsymbol{N}, \boldsymbol{M})$ is R-well-spaced. We say a box $\mathfrak{B}^n(\boldsymbol{N}, \boldsymbol{M})$ contains a (k-dimensional) R-well-spaced box if there is a set of k integers $1 \leq l_1 < \cdots < l_k \leq n$ such that $\prod_{i=1}^{k} \mathfrak{B}^1(N_{l_i}, M_{l_i})$ is an R-well-spaced box. For such a box, we are able to split up the product (5.12) into

$$\mathfrak{F}^n(\boldsymbol{N}, \boldsymbol{M}, \boldsymbol{\alpha}) = \mathfrak{F}^k_R(\boldsymbol{N'}, \boldsymbol{M'}, \boldsymbol{\alpha})\mathfrak{F}^{n-k}(\boldsymbol{N''}, \boldsymbol{M''}, \boldsymbol{\alpha}).$$

The choice of $\boldsymbol{N'}, \boldsymbol{M'}, \boldsymbol{N''}, \boldsymbol{M''}$ may of course not be unique.

Other than the initial Diophantine equation (5.5), we need to consider two more related systems of equations. The first system of equations is

$$\sum_{i=1}^{k}(x_i^j - y_i^j) + \sum_{i=1}^{s-k}(u_i^j - v_i^j) = 0, \quad (j = 1, \ldots, k),$$

where $\boldsymbol{x}, \boldsymbol{y}$ are tuples inside an R-well-spaced box $\mathfrak{B}^k(\boldsymbol{N}, M)$ with $M \geq 1$, $\boldsymbol{u}, \boldsymbol{v}$ are tuples inside a box $\mathfrak{B}^{s-k}(\xi, P)$ for some $\xi \in [-\frac{1}{2}, \frac{1}{2}]$ and $P \geq 1$, and furthermore

$\mathfrak{B}^k(\boldsymbol{N}, M) \times \mathfrak{B}^{s-k}(\xi, P) \subseteq]Q, Q+X]^s$ for some Q. Note that this forces $-X \le Q \le 0$ as $\mathbf{0} \in \mathfrak{B}^{s-k}(\xi, P)$. The corresponding counting integral is

$$I_R(\boldsymbol{N}, M, \xi, P) = \int_{[0,1[^k} |\mathfrak{F}_R^k(\boldsymbol{N}, M, \boldsymbol{\alpha})|^2 |f(\xi, P, \boldsymbol{\alpha})|^{2(s-k)} d\boldsymbol{\alpha}.$$

Let $I_R(M, P)$ denote the maximal number of solutions to the system of equations that occurs for any admissible ξ and $\boldsymbol{N}$ given R, M, P. Note that $I_R(M, P)$ is certainly bounded by $J_{s,k}(X+1)$ (by considering any solution $(\boldsymbol{x}, \boldsymbol{u}), (\boldsymbol{y}, \boldsymbol{v}) \in]Q, Q+X]^s$ and using Lemma 5.3.3) and is an integer, therefore well-defined.

The second supplementary system of equations is

$$\sum_{i=1}^{k} (x_i^j - y_i^j) + \sum_{i=1}^{k} (w_i^j - z_i^j) + \sum_{i=1}^{m-k} (u_i^j - v_i^j) + \sum_{i=1}^{s-m-k} (p_i^j - q_i^j) = 0, \quad (j = 1, \dots, k),$$

where $\boldsymbol{x}, \boldsymbol{y}$ are tuples inside an R-well-spaced box $\mathfrak{B}^k(\boldsymbol{N}, M)$ with $M \ge 1$, $\boldsymbol{w}, \boldsymbol{z}$ are tuples inside an R'-well-spaced box $\mathfrak{B}^k(\boldsymbol{N'}, L)$ with $L \ge 1$, $\boldsymbol{u}, \boldsymbol{v}$ are tuples inside a box $\mathfrak{B}^{m-k}(N'', L)$, $\boldsymbol{p}, \boldsymbol{q}$ are tuples inside a box $\mathfrak{B}^{s-m-k}(\xi, P)$ for some $\xi \in [-\frac{1}{2}, \frac{1}{2}]$ and $P \ge 1$, and furthermore $\mathfrak{B}^k(\boldsymbol{N'}, L) \subseteq \mathfrak{B}^k(\xi, P)$, $\mathfrak{B}^{m-k}(N'', L) \subseteq \mathfrak{B}^{m-k}(\xi, P)$ and $\mathfrak{B}^k(\boldsymbol{N}, M) \times \mathfrak{B}^{s-k}(\xi, P) \subseteq]Q, Q+X]^s$ for some Q. The corresponding counting integral is

$$\begin{aligned} &K_{R,R';m}(\boldsymbol{N}, M, \boldsymbol{N'}, L, N'', \xi, P) \\ &\quad = \int_{[0,1[^k} |\mathfrak{F}_R^k(\boldsymbol{N}, M, \boldsymbol{\alpha})|^2 |\mathfrak{F}_{R'}^k(\boldsymbol{N'}, L, \boldsymbol{\alpha})|^2 |f(N'', L, \boldsymbol{\alpha})|^{2(m-k)} |f(\xi, P, \boldsymbol{\alpha})|^{2(s-m-k)} d\boldsymbol{\alpha}. \end{aligned}$$

Let $K_{R,R';m}(M, P, L)$ denote the maximal number of solutions to the system of equations that occurs for any admissible $\xi, \boldsymbol{N}, \boldsymbol{N'}, N''$ given R, R', m, M, P, L. Again, this is well-defined.

As only very special types of these two integrals appear, we will shorten our notation to

$$\begin{aligned} I_{a,b}^g(X) &= I_{2^{-g}X^{1-a\theta}}(2^{-g}X^{1-a\theta}, X^{1-b\theta}), \\ K_{a,b;m}^{g,h}(X) &= K_{2^{-g}X^{1-a\theta}, 2^{-h}X^{1-b\theta}; m}(2^{-g}X^{1-a\theta}, X^{1-b\theta}, 2^{-h}X^{1-b\theta}), \end{aligned}$$

where θ is a sufficiently small parameter, taking on the role already mentioned in the introduction. The parameters g and h indicate the well-spacedness of our boxes and are very important for the argument given in the introduction.

The process of getting better and better upper bounds is of iterative nature, where in each step we decrease the exponent by a tiny bit. In every iteration, we will make use of previous upper bounds. Thus, we assume we have a bound of the shape

$$J_{s,k}(X) \le C \log_2(2X)^\delta X^{2s - \frac{1}{2}k(k+1) + \eta}, \tag{5.13}$$

with $0 \le \delta$ and $0 < \eta \le \frac{1}{2}k(k+1)$. Here, $\log_2$ denotes the logarithm to the base 2. Note that we certainly have such a bound with $(C, \delta, \eta) = (1, 0, \frac{1}{2}k(k+1))$.

To simplify calculations, we introduce the following normalisations:

$$\begin{aligned} [\![J_{s,k}(X)]\!] &= \frac{J_{s,k}(X)}{C \log_2(2X)^\delta X^{2s-\frac{1}{2}k(k+1)+\eta}}, \\ [\![I^g_{a,b}(X)]\!] &= \frac{I^g_{a,b}(X)}{C \log_2(2X)^\delta (X^{1-a\theta})^{2k-\frac{1}{2}k(k+1)} (X^{1-b\theta})^{2(s-k)} X^\eta}, \\ [\![K^{g,h}_{a,b;m}(X)]\!] &= \frac{K^{g,h}_{a,b;m}(X)}{C \log_2(2X)^\delta (X^{1-a\theta})^{2k-\frac{1}{2}k(k+1)} (X^{1-b\theta})^{2(s-k)} X^\eta}. \end{aligned} \tag{5.14}$$

Our assumed upper bound (5.13) is now reduced to the inequality $[\![J_{s,k}(X)]\!] \le 1$, which we will make use of rather frequently. To further simplify our proof, we adopt the rather unusual convention that

$$\sum^{W}$$

denotes a sum of at most W terms. In each term, the variables

$$\boldsymbol{N}, \boldsymbol{N'}, \boldsymbol{N''}, \boldsymbol{N'''}, \boldsymbol{N''''}, N, N', N_i, N'_i, \boldsymbol{U}, \boldsymbol{U'}, \boldsymbol{V}, \xi$$

may vary. Though, they are still required to satisfy certain properties coming from the context. These properties include, but are not limited to, ones such as 'being R-well-spaced' and 'being contained in a box of the shape $]Q, Q+X]^s$', and should always be clear from the context.

5.3.2 *Outline of the Proof*

To give the reader a better understanding of the whole argument, we give an overview of what is going on. Recall our assumption (5.13) and our normalisation (5.14). We have

$$[\![J_{s,k}(X)]\!] \le 1$$

and if $\eta > 0$ we would like to show

$$[\![J_{s,k}(X)]\!] \ll_{s,k} X^{-\Delta}$$

for some $\Delta > 0$ as large as possible. In a first step, we need to ensure that our variables are well-spaced. Secondly, we need to start the extraction, by making some variables small. Proposition 5.3.12 does both of these things and essentially gives

$$[\![J_{s,k}(X)]\!] \ll_{s,k,g} \log_2(2X) [\![I^g_{0,1}(X)]\!]. \tag{5.15}$$

Before extracting information, it is better to pre-well-space some variables for further extraction. This is done by Proposition 5.3.13, giving essentially

$$[\![I^g_{a,b}(X)]\!] \ll_{s,k,g,h,m} \log_2(2X)[\![K^{g,h}_{a,b;m}(X)]\!]. \tag{5.16}$$

Now that everything is prepared, we can extract some information whilst gaining something in the exponent. This is done by the argument given in the introduction (see Proposition 5.3.15 for details). This gives

$$[\![K^{g,h}_{a,b;m}(X)]\!] \ll_{s,k,g,h,m} X^{-\eta\frac{s-2k}{s-k}b\theta}[\![I^h_{b,kb}(X)]\!]^{\frac{k}{s-k}}. \tag{5.17}$$

In the end, we don't need to pre-well-space any more as it will be the last extraction. After the extraction, we bound the number of solutions trivially in terms of $J_{s,k}$. This is done in Proposition 5.3.16, giving the inequality

$$[\![I^g_{a,b}(X)]\!] \ll_{s,k,g} X^{-\eta\frac{s+k^2-k}{s}b\theta}X^{\frac{k^2(k^2-1)}{2s}b\theta}. \tag{5.18}$$

The idea is now to iterate through (5.16) and (5.17) as much as possible having fixed θ. By doing so, we see that the h cropping up in (5.16) will become the new g after (5.17) in the next iteration of (5.16). Thus, we'll get a sequence $\boldsymbol{g}$ on which the implied constants will depend. Moreover, we see that the pair (a,b) goes through the sequence $(0,1),(1,k),(k,k^2),\ldots,(k^{D-1},k^D)$. For simplicity, let us denote this sequence $(a_0,b_0),\ldots,(a_D,b_D)$. It turns out that to go through this many iterations one needs $\theta\le k^{-(D+1)}$. So, let us fix $\theta=k^{-(D+1)}$ and write

$$[\![J_{s,k}(X)]\!] = \frac{[\![J_{s,k}(X)]\!]}{[\![I^{g_0}_{0,1}(X)]\!]}\prod_{n=0}^{D-1}\left(\frac{[\![I^{g_n}_{a_n,b_n}(X)]\!]}{[\![I^{g_{n+1}}_{a_{n+1},b_{n+1}}(X)]\!]^{\frac{k}{s-k}}}\right)^{\left(\frac{k}{s-k}\right)^n}[\![I^{g_D}_{a_D,b_D}(X)]\!]^{\left(\frac{k}{s-k}\right)^D}.$$

By inserting the equations (5.15),(5.16),(5.17), and (5.18), we get

$$\begin{aligned}[\![J_{s,k}(X)]\!] &\ll_{s,k,\boldsymbol{g},\boldsymbol{m}} \log_2(2X)\prod_{n=0}^{D-1}\left(\log_2(2X)X^{-\eta\frac{s-2k}{s-k}k^n\theta}\right)^{\left(\frac{k}{s-k}\right)^n}\\ &\qquad\cdot\left(X^{-\eta\frac{s+k^2-k}{s}k^D\theta}X^{\frac{k^2(k^2-1)}{2s}k^D\theta}\right)^{\left(\frac{k}{s-k}\right)^D}\\ &\ll_{s,k,\boldsymbol{g},\boldsymbol{m}} \log_2(2X)^{\frac{2s-3k}{s-2k}}\left(X^\theta\right)^{\frac{k^2(k^2-1)}{2s}\left(\frac{k^2}{s-k}\right)^D-\eta\frac{s-2k}{s-k}\sum_{n=0}^D\left(\frac{k^2}{s-k}\right)^n},\end{aligned}$$

where we have made use of the trivial inequality $\frac{s+k^2-k}{s}\ge\frac{s-2k}{s-k}$ to make things simpler. Furthermore, we have extended the product to infinity to bound the exponent of the logarithm. It is now evident that if $s\ge k^2+k$ and $\eta>0$ we are able to find a sufficiently large D, such that the exponent is negative. This is of course only true if we can find a suitable choice of $\boldsymbol{g}$ and $\boldsymbol{m}$.

5.3.3 *Preliminaries*

In this section, we collect all lemmata which are needed to prove the core propositions in the next section. The first lemma is essential in almost every step and we will refer to it as the integer translation invariance.

Lemma 5.3.3 (Integer translation invariance). *For $l \in \mathbb{Z}$, we have*

$$\int_{[0,1[^k} \mathfrak{F}^s(\boldsymbol{N}, \boldsymbol{M}, \boldsymbol{\alpha})\mathfrak{F}^s(\boldsymbol{N'}, \boldsymbol{M'}, -\boldsymbol{\alpha})d\boldsymbol{\alpha} = \int_{[0,1[^k} \mathfrak{F}^s(\boldsymbol{N}+l, \boldsymbol{M}, \boldsymbol{\alpha})\mathfrak{F}^s(\boldsymbol{N'}+l, \boldsymbol{M'}, -\boldsymbol{\alpha})d\boldsymbol{\alpha}.$$

Proof. The first integral is counting the number of integer solutions to

$$\sum_{i=1}^{s} x_i^j = \sum_{i=1}^{s} y_i^j \quad (j = 1, \ldots, k), \tag{5.19}$$

with $\boldsymbol{x} \in \mathfrak{B}^s(\boldsymbol{N}, \boldsymbol{M})$ and $\boldsymbol{y} \in \mathfrak{B}^s(\boldsymbol{N'}, \boldsymbol{M'})$. By the Binomial Theorem, the system of equations (5.19) is equivalent to

$$\sum_{i=1}^{s}(x_i + l)^j = \sum_{i=1}^{s}(y_i + l)^j \quad (j = 1, \ldots, k),$$

with $\boldsymbol{x} + l \in \mathfrak{B}^s(\boldsymbol{N} + l, \boldsymbol{M})$ and $\boldsymbol{y} + l \in \mathfrak{B}^s(\boldsymbol{N'} + l, \boldsymbol{M'})$. This is exactly the corresponding Diophantine equation of the second integral and we have shown that translating by l gives a one to one correspondence between the two. Thus, the number of solutions are equal. □

Lemma 5.3.4. *For $x \geq 1$, we have*

$$\sqrt{2\pi}x^{x+\frac{1}{2}}e^{-x} \leq \Gamma(x+1) \leq ex^{x+\frac{1}{2}}e^{-x}.$$

Proof. Despite there being a vast literature on inequalities involving the Gamma function, the author was unable to find a reference for the above inequality, therefore we provide a proof. Consider the function $f(x) = \log(\Gamma(x+1)) - (x+\frac{1}{2})\log(x) + x$. From [Mer96], we know that

$$f''(x) = \sum_{k=1}^{\infty} \frac{1}{(k+x)^2} - \frac{1}{x} + \frac{1}{2x^2} > \frac{1}{6x^3} - \frac{1}{30x^5} > 0, \quad (x \geq 1).$$

Thus, $f(x)$ is convex for $x \geq 1$. Moreover, we have $f'(1) = \frac{1}{2} - \gamma < 0$, where γ is the Euler–Mascheroni constant, and

$$\lim_{x\to\infty} f(x) = \frac{1}{2}\log(2\pi)$$

from Stirling's approximation. Since $f(x)$ is convex, it follows that

$$\lim_{x\to\infty} f'(x) = 0.$$

Again, from the convexity, it follows that $f'(x) < 0$ for $x \geq 1$. Hence, the maximum is attained at $x = 1$ and the minimum at infinity. This gives the desired inequality. □

Lemma 5.3.5. *Let $S > 0$ be a real number. Further, let $1 \leq u \leq u + r - 1$ and let $\mathfrak{B}^r(\boldsymbol{N}, M)$ be an R-well-spaced box with $N_1 \leq N_2 \leq \cdots \leq N_r$. Suppose we are given two real r-tuples $\boldsymbol{x}, \boldsymbol{y} \in \mathfrak{B}^r(\boldsymbol{N}, M)$ with $-X < \boldsymbol{x}, \boldsymbol{y} \leq X$ such that*

$$\left|\sum_{i=1}^{r} x_i^j - \sum_{i=1}^{r} y_i^j\right| \leq SX^{j-1}$$

holds for every $j = u, \ldots, u + r - 1$. If $u > 1$ we furthermore need the assumption $x_j y_j > 0$ for $j = 1, \ldots, r - 1$ as well as $|x_r|, |y_r| \geq U > 0$. Then, we have

$$|x_r - y_r| \leq \sqrt{2}\left(\frac{e}{r}\right)^r \left(\frac{X}{R}\right)^{r-1} \left(\frac{X}{U}\right)^{u-1} S.$$

Proof. We follow Hua's argument quite closely (See [Hua65, Lemma 1, pp. 181-183]). We write

$$\sum_{i=1}^{r} \frac{x_i^j - y_i^j}{x_i - y_i} \cdot (x_i - y_i) = \theta_j X^{j-1}, \quad u \leq j \leq u + r - 1,$$

where $|\theta_j| \leq S$ for every $u \leq j \leq u + r - 1$. We regard this as linear system of equations in $x_1 - y_1, \ldots, x_r - y_r$. By Cramer's rule, we have

$$\Delta(x_r - y_r) - \Delta' = 0, \tag{5.20}$$

where

$$\Delta = \begin{vmatrix} \frac{x_1^u - y_1^u}{u(x_1 - y_1)} & \cdots & \frac{x_r^u - y_r^u}{u(x_r - y_r)} \\ \vdots & & \vdots \\ \frac{x_1^{u+r-1} - y_1^{u+r-1}}{(u+r-1)(x_1 - y_1)} & \cdots & \frac{x_r^{u+r-1} - y_r^{u+r-1}}{(u+r-1)(x_r - y_r)} \end{vmatrix},$$

$$\Delta' = \begin{vmatrix} \frac{x_1^u - y_1^u}{u(x_1 - y_1)} & \cdots & \frac{x_{r-1}^u - y_{r-1}^u}{u(x_{r-1} - y_{r-1})} & \frac{\theta_u}{u} X^{u-1} \\ \vdots & & \vdots & \vdots \\ \frac{x_1^{u+r-1} - y_1^{u+r-1}}{(u+r-1)(x_1 - y_1)} & \cdots & \frac{x_{r-1}^{u+r-1} - y_{r-1}^{u+r-1}}{(u+r-1)(x_{r-1} - y_{r-1})} & \frac{\theta_{u+r-1}}{u+r-1} X^{u+r-2} \end{vmatrix}.$$

Now, one can rewrite (5.20) as

$$\frac{1}{\prod_{i=1}^{r}(x_i - y_i)} \int_{y_1}^{x_1} \cdots \int_{y_r}^{x_r} \left(\Delta_{u,r}(x_r - y_r) - \Delta'_{u,r}\right) dz_1 \dots dz_r = 0,$$

where

$$\Delta_{u,r} = \begin{vmatrix} z_1^{u-1} & \dots & z_r^{u-1} \\ \vdots & & \vdots \\ z_1^{u+r-2} & \dots & z_r^{u+r-2} \end{vmatrix},$$

$$\Delta'_{u,r} = \begin{vmatrix} z_1^{u-1} & \dots & z_{r-1}^{u-1} & \frac{\theta_u}{u} X^{u-1} \\ \vdots & & \vdots & \vdots \\ z_1^{u+r-2} & \dots & z_{r-1}^{u+r-2} & \frac{\theta_{u+r-1}}{u+r-1} X^{u+r-2} \end{vmatrix}.$$

In the case of $x_i = y_i$ for some i, we can still make sense of the above argument in terms of limits, which do exist. By the mean value theorem of integral calculus, there is a choice of $z_i \in [x_i, y_i]$ for $1 \leq i \leq r$ such that

$$\Delta_{u,r}(x_r - y_r) - \Delta'_{u,r} = 0.$$

By considering Vandermonde determinants, we find the identity

$$\Delta_{u,r} = \Delta_{u,r-1} \cdot z_r^{u-1} \prod_{i=1}^{r-1} (z_r - z_i)$$

and moreover

$$\Delta_{u,r-1} \neq 0$$

as the z_i are pairwise different and in the case $u > 1$ we also have $z_i \neq 0$ as $0 \notin [x_i, y_i]$ for $i = 1, \dots, r-1$. Let us denote the elementary symmetric polynomial of degree $r - i$ in the variables $z_1, \dots, z_{r-1}$ by σ_{r-i}. These satisfy $|\sigma_{r-i}| \leq \binom{r-1}{r-i} X^{r-i}$ as $-X \leq \boldsymbol{z} \leq X$. Therefore, in the expansion of $\Delta_{u,r}$, the absolute values of the coefficient of z_r^{u+i-2} are equal to

$$|\sigma_{r-i}\Delta_{u,r-1}| \leq \binom{r-1}{r-i} X^{r-i} |\Delta_{u,r-1}|.$$

By using the column minor of expansion of $\Delta'_{u,r}$ and comparing it with the corresponding one of $\Delta_{u,r}$, we find that

$$\begin{aligned} |\Delta'_{u,r}| &\leq |\Delta_{u,r-1}| \sum_{i=1}^{r} \frac{|\sigma_{r-i}||\theta_{u+i-1}|}{u+i-1} X^{u+i-2} \leq |\Delta_{u,r-1}| S X^{u+r-2} \sum_{i=1}^{r} \frac{1}{u+i-1} \binom{r-1}{r-i} \\ &\leq \frac{2^r}{r} \cdot |\Delta_{u,r-1}| \cdot S X^{u+r-2}, \end{aligned}$$

since

$$\sum_{i=1}^{r} \frac{1}{u+i-1}\binom{r-1}{r-i} \le \sum_{i=1}^{r} \frac{1}{i}\binom{r-1}{r-i} = \sum_{i=1}^{r} \frac{1}{r}\binom{r}{r-i} \le \frac{2^r}{r}.$$

It follows that

$$|x_r - y_r| \le \frac{2^r \cdot SX^{u+r-2}}{r \cdot |z_r|^{u-1} \prod_{i=1}^{r-1}(z_r - z_i)} \le \frac{2^r}{r \cdot \prod_{i=1}^{r-1}(2i-1)} \left(\frac{X}{R}\right)^{r-1} \left(\frac{X}{U}\right)^{u-1} S,$$

where we have used $|z_r - z_{r-i}| \ge (2i-1)R$ and $|z_r| \ge U$. Furthermore, we have for $r \ge 3$

$$\begin{aligned}\prod_{i=1}^{r-1}(2i-1) &= \frac{\Gamma(2r-1)}{\Gamma(r)\cdot 2^{r-1}} = \frac{2^{r-1}}{\sqrt{\pi}}\Gamma\left(r-\frac{1}{2}\right) \\ &\ge 2^{r-\frac{1}{2}}\left(r-\frac{3}{2}\right)^{r-1} e^{\frac{3}{2}-r} \\ &\ge 2^{r-\frac{1}{2}} r^{r-1} e^{-r},\end{aligned}$$

where we have made use of Lemma 5.3.4. It is easily verified, that this inequality also holds for $r = 2$. Thus, we have

$$|x_r - y_r| \le \sqrt{2}\left(\frac{e}{r}\right)^r \left(\frac{X}{R}\right)^{r-1} \left(\frac{X}{U}\right)^{u-1} S,$$

which holds also for $r = 1$ for trivial reasons. □

Lemma 5.3.6. *We have for $r \ge 1$:*

$$\prod_{n=1}^{r} n^{n-1} \ge r^{\frac{1}{2}r(r-2)} e^{-\frac{1}{4}(r-1)(r-3)}.$$

Proof. The function $(x-1)\log(x)$ is convex with a minimum of 0 at 1. Thus,

$$\begin{aligned}\prod_{n=1}^{r} n^{n-1} &= \exp\left(\sum_{n=1}^{r}(n-1)\log(n)\right) \\ &\ge \exp\left(\int_1^r (x-1)\log(x)dx\right) \\ &= r^{\frac{1}{2}r(r-2)} e^{-\frac{1}{4}(r-1)(r-3)}.\end{aligned}$$

□

Lemma 5.3.7. *Let $1 \le r \le k$ and furthermore let $\mathfrak{B}^r(\boldsymbol{N}, M) \subseteq]-X, X]^r$ be an R-well-spaced box with $N_1 \le N_2 \le \cdots \le N_r$ and $M, S \ge 1$. Assume as well $R \le X/(2k)$. Let $\mathcal{Z}_W(\mathfrak{B}^r(\boldsymbol{N}, M), \boldsymbol{U})$ be the number of integer solutions $\boldsymbol{x} \in \mathfrak{B}^r(\boldsymbol{N}, M)$ counted with multiplicity $W(\boldsymbol{x}) \ge 0$ satisfying*

$$\sum_{i=1}^{r} x_i^j \in U_j \quad (j = 1, \ldots, r), \tag{5.21}$$

where U_j is an interval of size at most SX^{j-1}. Then, we have the bound

$$\mathcal{Z}_W(\mathfrak{B}^r(\boldsymbol{N}, M), \boldsymbol{U}) \leq 2^{\frac{1}{2}r(r+1)} e^{\frac{1}{4}(3r+1)(r-1)} r^{-\frac{1}{2}r(r-2)} \cdot \left(\frac{X}{R}\right)^{\frac{1}{2}r(r-1)} \cdot \mathcal{Z}_W(\mathfrak{B}^r(\boldsymbol{N'}, \boldsymbol{S'}), \boldsymbol{U}),$$

for some sub-box $\mathfrak{B}^r(\boldsymbol{N'}, \boldsymbol{S'})$ of $\mathfrak{B}^r(\boldsymbol{N}, M)$ with $1 \leq \boldsymbol{S'} \leq S$.

Remark 5.3.8. If $S \leq M$, then we are of course able to choose $\boldsymbol{S'} \equiv S$.

Proof. This will follow from the inequality

$$\mathcal{Z}_W(\mathfrak{B}^r(\boldsymbol{N}, M), \boldsymbol{U}) \leq 2^{\frac{1}{2}r(r+1)} e^{\frac{1}{2}r(r+1)-1} \prod_{n=1}^{r} n^{-(n-1)} \cdot \left(\frac{X}{R}\right)^{\frac{1}{2}r(r-1)} \cdot \mathcal{Z}_W(\mathfrak{B}^r(\boldsymbol{N'}, \boldsymbol{S'}), \boldsymbol{U}), \tag{5.22}$$

which we shall prove inductively, and Lemma 5.3.6. The inequality (5.22) clearly holds for $r = 1$. Thus, we may assume $r \geq 2$ from now on. Without loss of generality, we may assume $W(\boldsymbol{x}) = 0$ if $\boldsymbol{x}$ does not satisfy (5.21). If there are no solutions, then (5.22) holds trivially. Assume now there is a solution $\boldsymbol{x}$. If we have another solution $\boldsymbol{x'}$ we deduce

$$\left| \sum_{i=1}^{r} x_i^j - \sum_{i=1}^{r} x_i'^j \right| \leq SX^{j-1}$$

from (5.21). We can apply Lemma 5.3.5 with $u = 1$ to deduce that

$$|x_r - x_r'| \leq \sqrt{2} \left(\frac{e}{r}\right)^r \left(\frac{X}{R}\right)^{r-1} S.$$

Thus, we can find a sub-box of $\mathfrak{B}^1(N_r, M)$ of size at most

$$\sqrt{2} \left(\frac{e}{r}\right)^r \left(\frac{X}{R}\right)^{r-1} S + 1 \leq 2 \left(\frac{e}{r}\right)^r \left(\frac{X}{R}\right)^{r-1} S$$

in which all the x_r's lie, as

$$\left(\frac{X}{R}\right)^{r-1} \left(\frac{e}{r}\right)^r \geq (2k)^{r-1} \left(\frac{e}{r}\right)^r \geq \frac{1}{2r}(2e)^r \geq e$$

and $\sqrt{2} + e^{-1} \leq 2$. This box we may further split up into at most

$$2 \cdot (r-1) \left(\frac{e}{r}\right)^r \left(\frac{X}{R}\right)^{r-1} + 1 \leq 2r \left(\frac{e}{r}\right)^r \left(\frac{X}{R}\right)^{r-1}$$

boxes of size at most $\frac{S}{r-1}$. We now consider the interval S_r', say, which contributes the most towards $\mathcal{Z}_W(\mathfrak{B}^r(\boldsymbol{N}, M), \boldsymbol{U})$. By assumption, we still have at least a solution $\boldsymbol{x}$.

Consider now a second solution $\boldsymbol{x'}$. Call $\boldsymbol{y}$ and $\boldsymbol{y'}$ their restrictions to the first $k-1$ coordinates. Note that we have

$$\begin{aligned}\left|\sum_{i=1}^{r-1} y_i^j - \sum_{i=1}^{r-1} y_i'^j\right| &\leq \left|\sum_{i=1}^{r} x_i^j - \sum_{i=1}^{r} x_i'^j\right| + |x_r^j - x_r'^j| \\ &\leq SX^{j-1} + \frac{S}{r-1}\cdot jX^{j-1} \\ &\leq 2SX^{j-1}\end{aligned}$$

for $j = 1, \ldots, r-1$. Thus, their j-th power sum is contained in some interval of length at most $2SX^{j-1}$. Each of these intervals we split into half, yielding

$$\mathcal{Z}_W(\mathfrak{B}^r(\boldsymbol{N}, M), \boldsymbol{U}) \leq 2r\left(\frac{e}{r}\right)^r \left(\frac{X}{R}\right)^{r-1} \cdot \sum^{2^{r-1}} \mathcal{Z}_{W'}(\mathfrak{B}^{r-1}(\boldsymbol{N}, M), \boldsymbol{U'}),$$

where $W'(\boldsymbol{y}) = \sum_{x_r \in S_r'} W((\boldsymbol{y}, x_r))$. We now take the maximum and apply the induction hypothesis for $r-1$ with W'. The induction is now complete as

$$\begin{aligned}2r\left(\frac{e}{r}\right)^r \left(\frac{X}{R}\right)^{r-1} \cdot 2^{r-1} \cdot 2^{\frac{1}{2}r(r-1)} e^{\frac{1}{2}r(r-1)-1} \prod_{n=1}^{r-1} n^{-(n-1)} \cdot \left(\frac{X}{R}\right)^{\frac{1}{2}(r-1)(r-2)} \\ = 2^{\frac{1}{2}r(r+1)} e^{\frac{1}{2}r(r+1)-1} \prod_{n=1}^{r} n^{-(n-1)} \cdot \left(\frac{X}{R}\right)^{\frac{1}{2}r(r-1)}.\end{aligned}$$

and

$$\mathcal{Z}_{W'}(\mathfrak{B}^{r-1}(\boldsymbol{N'}, \boldsymbol{S'}), \boldsymbol{U'}) \leq \mathcal{Z}_W(\mathfrak{B}^r(\boldsymbol{N'}, \boldsymbol{S'}), \boldsymbol{U}),$$

where $\mathfrak{B}^r(\boldsymbol{N'}, \boldsymbol{S'}) = \mathfrak{B}^{r-1}(\boldsymbol{N'}, \boldsymbol{S'}) \times S_r'$. □

Lemma 5.3.9. *Let $k, m, D \in \mathbb{N}$ with $m \geq k \geq 2$. The set of integers $(d_1, \ldots, d_m)$ with $0 \leq d_i < D$ for $i = 1, \ldots, m$ is said to contain a well-spaced (k-dimensional) sub-tuple if there are k of them, say $d_{i_1}, \ldots, d_{i_k}$, satisfying*

$$d_{i_{j+1}} - d_{i_j} > 1 \quad j = 1, \ldots, k-1.$$

The number of tuples not containing a well-spaced sub-tuple is bounded by

$$2^m k^m D^{k-1}.$$

Proof. See [Hua65, Lemma 4.3]. There is however a slight error in the proof. Their argument gives the bound

$$\sum_{u=1}^{k-1} \binom{D}{u} (2u)^m \leq 2^m (k-1)^m D^{k-1} \sum_{u=1}^{k-1} \frac{1}{u!}$$
$$\leq 2^m k^m e^{-\frac{m}{k}} D^{k-1}(e-1)$$
$$\leq 2^m k^m D^{k-1}.$$

If one works a little bit harder one may also recover the bound claimed in [Hua65]. □

This next lemma is the key to the well-spacing Propositions 5.3.12 and 5.3.13. It is essentially about bounding the number of solutions in a box $\mathfrak{B}^m(N,P) \times \mathfrak{B}^m(N,P)$ by the number of solutions in its sub-boxes, most of which contain a (k-dimensional) R-well-spaced box with R being of large size compared to the sub-box itself.

Lemma 5.3.10. *For $G \geq 1$, $m \geq k+1, k \geq 2$ and $P \geq 2^G$, we have that $|f(N,P,\boldsymbol{\alpha})|^{2m}$ is bounded by*

$$G\Bigg[\sum_{g=\lceil \log_2(2k) \rceil}^{G} \frac{2^m L_{g-1}}{m-k} \sum^{(m-k)2^m L_{g-1}} |\mathfrak{F}^k_{2^{-g}P}(\boldsymbol{N'}, 2^{-g}P, \boldsymbol{\alpha})|^2 |f(N', 2^{-g}P, \boldsymbol{\alpha})|^{2(m-k)}$$
$$+ \frac{L_G}{m} \sum^{mL_G} |f(N', 2^{-G}P, \boldsymbol{\alpha})|^{2m}\Bigg],$$

where all the boxes on the right-hand side are contained in the box $\mathfrak{B}^{2m}(N,P)$ and

$$L_g = 2^m k^m \cdot (2^g)^{k-1}.$$

Proof. At the heart of the argument lies the equality

$$f(N,P,\boldsymbol{\alpha}) = f(N-P/4, P/2, \boldsymbol{\alpha}) + f(N+P/4, P/2, \boldsymbol{\alpha}). \tag{5.23}$$

Iterating this equality shows that we have

$$f(N,P,\boldsymbol{\alpha}) = \sum_{d=0}^{2^g-1} f(N - P/2 + (d+1/2)2^{-g}P, 2^{-g}P, \boldsymbol{\alpha})$$

for every $g \in \mathbb{N}$. This further leads to

$$f(N,P,\boldsymbol{\alpha})^m = \sum_{0 \leq \boldsymbol{d} \leq 2^g-1} \prod_{i=1}^{m} f(N - P/2 + (d_i+1/2)2^{-g}P, 2^{-g}P, \boldsymbol{\alpha}). \tag{5.24}$$

Now, the above box $\mathfrak{B}^m(N - P/2 + (\boldsymbol{d} + 1/2)2^{-g}P, 2^{-g}P)$ contains a (k-dimensional) $2^{-g}P$-well-spaced box if and only if the tuple $\boldsymbol{d} = (d_1, \ldots, d_m)$ contains a (k-dimensional) well-spaced sub-tuple.

Our plan is to extract the tuples which contain a well-spaced sub-tuple from (5.24) before using (5.23) on the remaining summands. For this purpose, we are going to use the binary expansion of the d's. Given $\boldsymbol{d} \in \mathbb{N}_0^m$, we define the predecessor of $\boldsymbol{d}$ to be $\boldsymbol{p}(\boldsymbol{d}) = (\lfloor d_1/2 \rfloor, \ldots, \lfloor d_m/2 \rfloor)$. We also define the set of successors of $\boldsymbol{d}$ as $S(\boldsymbol{d}) = \{\boldsymbol{d'} \in \mathbb{N}_0^m | \boldsymbol{p}(\boldsymbol{d'}) = \boldsymbol{d}\}$. Note, that if $\boldsymbol{p}(\boldsymbol{d})$ contains a (k-dimensional) well-spaced sub-tuple, then so does $\boldsymbol{d}$. We abbreviate '$\boldsymbol{d}$ contains a (k-dimensional) well-spaced sub-tuple' to '$\boldsymbol{d}$ is good'. We are now able to prove the following identity for $G \in \mathbb{N}_0$ inductively:

$$\begin{aligned} f(N, P, \boldsymbol{\alpha})^m = & \sum_{g=0}^{G} \sum_{\substack{0 \le \boldsymbol{d} \le 2^g - 1 \\ \boldsymbol{d} \text{ is good} \\ \boldsymbol{p}(\boldsymbol{d}) \text{ is not good}}} \prod_{i=1}^{m} f(N - P/2 + (d_i + 1/2)2^{-g}P, 2^{-g}P, \boldsymbol{\alpha}) \\ & + \sum_{\substack{0 \le \boldsymbol{d} \le 2^G - 1 \\ \boldsymbol{d} \text{ is not good}}} \prod_{i=1}^{m} f(N - P/2 + (d_i + 1/2)2^{-G}P, 2^{-G}P, \boldsymbol{\alpha}). \end{aligned} \tag{5.25}$$

For $G = 0$, the right hand-side is just $f(N, P, \boldsymbol{\alpha})^m$ as the first sum is empty. The induction step follows from the identity

$$\begin{aligned} \prod_{i=1}^{m} f(N - P/2 + & (d_i + 1/2)2^{-G}P, 2^{-G}P, \boldsymbol{\alpha}) \\ & = \sum_{\boldsymbol{d'} \in S(\boldsymbol{d})} \prod_{i=1}^{m} f(N - P/2 + (d_i' + 1/2)2^{-(G+1)}P, 2^{-(G+1)}P, \boldsymbol{\alpha}), \end{aligned}$$

which is just (5.23) applied to each factor, applied to the latter sum in (5.25); i.e. the $\boldsymbol{d}$'s which are not good, and splitting up into good and not good tuples.

Now, we note that if $g \le \lceil \log_2(2k) \rceil - 1$ we have that all tuples $0 \le \boldsymbol{d} \le 2^g - 1$ are not good, because if there were a good one, we would have

$$D = 2^g \ge 1 + d_{i_k} = 1 + \sum_{j=1}^{k-1} (d_{i_{j+1}} - d_{i_j}) + d_{i_1} \ge 1 + 2(k-1).$$

If $g \ge 1$ we have $2^g \ge 2k$ as $2k - 1$ is odd, leading to a contradiction. For $g = 0$, there are obviously no good tuples. The next thing we note is the number of not good tuples $0 \le \boldsymbol{d} \le 2^g - 1$ is at most

$$L_g = 2^m k^m \cdot (2^g)^{k-1},$$

by Lemma 5.3.9. Moreover, we have $|S(\boldsymbol{d})| = 2^m$ which shows that the set of tuples $0 \le \boldsymbol{d} \le 2^g - 1$ such that $\boldsymbol{d}$ is good and $\boldsymbol{p}(\boldsymbol{d})$ is not good has cardinality at most $2^m L_{g-1}$. Therefore, we conclude that the equality (5.25) is of the shape

$$\begin{aligned} f(N,P,\boldsymbol{\alpha})^m = & \sum_{g=\lceil \log_2(2k)\rceil}^{G} \sum^{2^m L_{g-1}} \mathfrak{F}^k_{2^{-g}P}(\boldsymbol{N}', 2^{-g}P, \boldsymbol{\alpha}) \mathfrak{F}^{m-k}(\boldsymbol{N}'', 2^{-g}P, \boldsymbol{\alpha}) \\ & + \sum^{L_G} \mathfrak{F}^m(\boldsymbol{N}''', 2^{-G}P, \boldsymbol{\alpha}). \end{aligned}$$

Applying Cauchy–Schwarz twice yields

$$\begin{aligned} |f(N,P,\boldsymbol{\alpha})|^{2m} \le & G\Bigg[\sum_{g=\lceil \log_2(2k)\rceil}^{G} \left(\sum^{2^m L_{g-1}} |\mathfrak{F}^k_{2^{-g}P}(\boldsymbol{N}', 2^{-g}P, \boldsymbol{\alpha})| |\mathfrak{F}^{m-k}(\boldsymbol{N}'', 2^{-g}P, \boldsymbol{\alpha})| \right)^2 \\ & + \left(\sum^{L_G} |\mathfrak{F}^m(\boldsymbol{N}''', 2^{-G}P, \boldsymbol{\alpha})| \right)^2 \Bigg] \\ \le & G\Bigg[\sum_{g=\lceil \log_2(2k)\rceil}^{G} 2^m L_{g-1} \sum^{2^m L_{g-1}} |\mathfrak{F}^k_{2^{-g}P}(\boldsymbol{N}', 2^{-g}P, \boldsymbol{\alpha})|^2 |\mathfrak{F}^{m-k}(\boldsymbol{N}'', 2^{-g}P, \boldsymbol{\alpha})|^2 \\ & + L_G \sum^{L_G} |\mathfrak{F}^m(\boldsymbol{N}''', 2^{-G}P, \boldsymbol{\alpha})|^2 \Bigg], \end{aligned}$$

since $G - \lceil \log_2(2k)\rceil + 1 + 1 \le G$. By further using the inequality between the arithmetic and geometric mean in the shape

$$|\mathfrak{F}^r(\boldsymbol{N}, M, \boldsymbol{\alpha})|^2 \le \frac{1}{r} \sum_{i=1}^{r} |f(N_i, M, \boldsymbol{\alpha})|^{2r},$$

we prove the desired inequality. □

Remark 5.3.11. Potentially, one could gain more $\log_2(2X)$ savings if one were to allow mixed terms with different g's, but the state of affairs is already complicated enough as it is and so we omit exploring this possibility.

5.3.4 *Core Propositions*

In this section, we prove the core propositions which will be used in the final argument, as outlined in Section 5.3.2. From now on, we will also assume that $k \ge 3$.

This first proposition well-spaces a set of variables in order to get the iteration started.

Proposition 5.3.12. *Let $G \in \mathbb{N}$ and assume $X^\theta \geq 2^G$ with $0 < \theta \leq \frac{1}{k^2}$. Furthermore, let $s \geq m \geq k+1$ and $m \leq 8k^2$. Then, we have that $[\![J_{s,k}(X)]\!]$ is bounded by the sum:*

$$C' \cdot G \sum_{g=\lceil \log_2(2k) \rceil}^{G} (2^g)^{2(k-1)-2(m-k)} [\![I^g_{0,1}(X)]\!] + C'' \cdot G \cdot \left(2^G\right)^{2(k-1)-\frac{m}{s}(2s-\frac{1}{2}k(k+1)+\eta)},$$

where

$$C' = 2^{6m-4k+2} k^{2m} \cdot \left(1 + \frac{1}{X^\theta}\right)^{2(s-m)},$$
$$C'' = 2^{2m+1} k^{2m}.$$

Proof. We will use Lemma 5.3.10 for $2m$ factors in $J_{s,k}(X)$. We find

$$J_{s,k}(X) \leq G \left[\sum_{g=\lceil \log_2(2k) \rceil}^{G} \frac{2^m L_{g-1}}{m-k} \sum^{(m-k)2^m L_{g-1}} \mathcal{W}_g + \frac{L_G}{m} \sum^{mL_G} \mathcal{N} \right],$$

where

$$\mathcal{W}_g = \int_{[0,1[^k} |\mathfrak{F}^k_{2^{-g}X}(\boldsymbol{N}, 2^{-g}X, \boldsymbol{\alpha})|^2 |f(N, 2^{-g}X, \boldsymbol{\alpha})|^{2(m-k)} |f(X/2, X, \boldsymbol{\alpha})|^{2(s-m)} d\boldsymbol{\alpha},$$
$$\mathcal{N} = \int_{[0,1[^k} |f(N, 2^{-G}X, \boldsymbol{\alpha})|^{2m} |f(X/2, X, \boldsymbol{\alpha})|^{2(s-m)} d\boldsymbol{\alpha}.$$

We refer to the first part as the well-spaced part and second part as the non-well-spaced part. Let us first consider the part which is non-well-spaced. There, we have to consider the integral $\mathcal{N}$. We find

$$\begin{aligned}\mathcal{N} \leq & \left(\int_{[0,1[^k} |f(N, 2^{-G}X, \boldsymbol{\alpha})|^{2s} d\boldsymbol{\alpha} \right)^{\frac{m}{s}} \left(\int_{[0,1[^k} |f(X/2, X, \boldsymbol{\alpha})|^{2s} d\boldsymbol{\alpha} \right)^{\frac{s-m}{s}} \\ \leq & J_{s,k}(2^{-G}X + 1)^{\frac{m}{s}} \cdot J_{s,k}(X)^{\frac{s-m}{s}}\end{aligned}$$

by Hölder's inequality and the integer translation invariance. Now, we have

$$2^{-G}X + 1 = 2^{-G}X(1 + 2^G X^{-1}) \leq 2^{-G}X \left(1 + \frac{9}{2^8 k^2}\right)$$

as

$$2^G X^{-1} \leq X^{-1+\theta} \leq 2^{\theta^{-1}(-1+\theta)} \leq 2^{1-k^2} \leq \frac{9}{2^8 k^2}.$$

In conclusion, the overall contribution to $J_{s,k}(X)$ from the non-well-spaced part is

$$G \cdot 2^{2m} k^{2m} \left(2^G\right)^{2(k-1)} \cdot J_{s,k}\left(\left(\left(1 + \frac{9}{2^8 k^2}\right) 2^{-G} X \right)^{\frac{m}{s}} \cdot J_{s,k}(X)^{\frac{s-m}{s}}, \quad (5.26)$$

where we recalled $L_G = 2^m k^m (2^G)^{k-1}$.

For the $2^{-g}X$-well-spaced ones, we have to consider the integral $\mathcal{W}_g$. We split up the box $\mathfrak{B}^1(N, 2^{-g}X)$, respectively $\mathfrak{B}^1(X/2, X)$, into boxes of size at most $X^{1-\theta}$. Moreover, we may assume all the boxes have size $X^{1-\theta}$ as it can only happen, that we account for solutions multiple times. There are at most

$$\frac{2^{-g}X}{X^{1-\theta}} + 1 = 2^{-g}X^{\theta} \cdot \left(1 + \frac{2^g}{X^{\theta}}\right) \leq 2^{-g}X^{\theta} \cdot 2 = T_g', \text{ say,}$$

respectively

$$\frac{X}{X^{1-\theta}} + 1 = X^{\theta}\left(1 + \frac{1}{X^{\theta}}\right) = T_g'', \text{ say,}$$

boxes of this kind. Thus, we have that $\mathcal{W}_g$ is bounded by

$$\int_{[0,1[^k} |\mathfrak{F}_{2^{-g}X}(\boldsymbol{N}, 2^{-g}X, \boldsymbol{\alpha})|^2 \left|\sum^{T_g'} f(N, X^{1-\theta}, \boldsymbol{\alpha})\right|^{2(m-k)} \left|\sum^{T_g''} f(N', X^{1-\theta}, \boldsymbol{\alpha})\right|^{2(s-m)} d\boldsymbol{\alpha},$$

which we immediately bound further by using Cauchy–Schwarz as follows

$$\begin{aligned}\left|\sum^{T_g'} f(N, X^{1-\theta}, \boldsymbol{\alpha})\right|^{2(m-k)} &\left|\sum^{T_g''} f(N', X^{1-\theta}, \boldsymbol{\alpha})\right|^{2(s-m)} \\ &\leq \left(T_g' \sum^{T_g'} |f(N, X^{1-\theta}, \boldsymbol{\alpha})|^2\right)^{m-k} \left(T_g'' \sum^{T_g''} |f(N', X^{1-\theta}, \boldsymbol{\alpha})|^2\right)^{s-m}.\end{aligned}$$

If we define

$$T_g = T_g'^{(m-k)} \cdot T_g''^{(s-m)} = 2^{m-k}\left(2^{-g}\right)^{m-k}\left(1 + \frac{1}{X^{\theta}}\right)^{s-m} X^{(s-k)\theta},$$

then we find after expanding the product of sums into a sum of products that

$$\mathcal{W}_g \leq T_g \int_{[0,1[^k} |\mathfrak{F}_{2^{-g}X}(\boldsymbol{N}, 2^{-g}X, \boldsymbol{\alpha})|^2 \left(\sum^{T_g} \prod_{i=1}^{s-k} |f(N_i'', X^{1-\theta}, \boldsymbol{\alpha})|^2\right) d\boldsymbol{\alpha}.$$

By using the inequality between the arithmetic and geometric mean and the integer translation invariance, we bound the above further by

$$\begin{aligned}\mathcal{W}_g \leq &\frac{T_g}{s-k} \sum^{(s-k)T_g} \int_{[0,1[^k} |\mathfrak{F}_{2^{-g}X}(\boldsymbol{N}, 2^{-g}X, \boldsymbol{\alpha})|^2 |f(N'', X^{1-\theta}, \boldsymbol{\alpha})|^{2(s-k)} d\boldsymbol{\alpha} \\ = &\frac{T_g}{s-k} \sum^{(s-k)T_g} \int_{[0,1[^k} |\mathfrak{F}_{2^{-g}X}(\boldsymbol{N'}, 2^{-g}X, \boldsymbol{\alpha})|^2 |f(\xi, X^{1-\theta}, \boldsymbol{\alpha})|^{2(s-k)} d\boldsymbol{\alpha} \\ \leq &T_g^2 \cdot I_{0,1}^g(X).\end{aligned}$$

This last step of translating the variables such that some of the variables are small is an essential prerequisite for the extraction argument (5.9) to follow. We find that the well-spaced contribution is at most

$$G \sum_{g=\lceil \log_2(2k) \rceil}^{G} 2^{6m-4k+2} k^{2m} \left(2^g\right)^{2(k-1)-2(m-k)} \left(1 + \frac{1}{X^\theta}\right)^{2(s-m)} X^{2(s-k)\theta} \cdot I_{0,1}^g(X). \quad (5.27)$$

In conclusion, we have that $J_{s,k}(X)$ is bounded by the sum of (5.26) and (5.27). We now normalise this inequality to get an inequality for $[\![J_{s,k}(X)]\!]$. After normalising, we easily find that the well-spaced part of the proposition is true. In the non-well-spaced part, we collect an additional factor of

$$\frac{\log_2\left(2\left(1+\frac{9}{4k^2}\right)2^{-G}X\right)^{\delta \frac{m}{s}}}{\log_2(2X)^{\delta \frac{m}{s}}} \left(2^{-G}\right)^{\frac{m}{s}(2s - \frac{1}{2}k(k+1)+\eta)} \left(1 + \frac{9}{2^8 k^2}\right)^{\frac{m}{s}(2s-\frac{1}{2}k(k+1)+\eta)}.$$

The fraction of log's is trivially bounded by 1 and since $\eta \leq \frac{1}{2}k(k+1)$ and $m \leq 8k^2$ we have

$$\left(1 + \frac{9}{2^8 k^2}\right)^{\frac{m}{s}(2s-\frac{1}{2}k(k+1)+\eta)} \leq \left(1 + \frac{9}{2^8 k^2}\right)^{2m} \leq e^{\frac{9m}{2^7 k^2}} \leq 2.$$

This concludes the proof. □

This next proposition is almost analogous to the previous one. The observant reader may notice an important difference though. Here, the well-spacing step takes place two steps ahead of when it is needed. This is advantageous as it allows for a smaller choice of the parameter m in Lemma 5.3.10.

Proposition 5.3.13. *Let $H, a, b \in \mathbb{N}_0$ with $H \geq 1$ and $b > a \geq 0$. Assume $\theta \in \mathbb{R}$ satisfies $1 \geq k^2 b\theta > 0$. Let $X \geq 2^{\theta^{-1}}$ and $X^{kb\theta} \geq 2^H$. Let $g \in \mathbb{N}$ such that $X^{b\theta} \geq 2^g \geq 2k$. Furthermore, let $800k \geq m \geq k+1$ and $6^6 k^2 \log(k) \geq s - k \geq m$. Then, we have that $[\![I_{a,b}^g(X)]\!]$ is bounded by the sum*

$$\begin{aligned} & C' \cdot H \sum_{h=\lceil \log_2(2k) \rceil}^{H} \left(2^h\right)^{2(k-1)} [\![K_{a,b;m}^{g,h}(X)]\!] \\ & + C'' \cdot H \left(2^g\right)^{-\frac{k}{s}(2s-\frac{1}{2}k(k+1)+\eta)} \left(2^H\right)^{2(k-1)-\frac{m}{s}(2s-\frac{1}{2}k(k+1)+\eta)} \\ & \qquad \cdot \left(X^\theta\right)^{\frac{1}{2}k(k+1)\frac{s-k}{s}(b-a)} \left(X^{-\eta\theta}\right)^{\frac{k}{s}a + \frac{s-k}{s}b}, \end{aligned}$$

where

$$C' = 2^{4m-2k+2} \cdot k^{2m},$$

$$C'' = 2^{2m+1} \cdot k^{2m+1}.$$

Remark 5.3.14. In due course, H and m will be chosen in such a way that we have a power saving in X in the non-well-spaced part.

Proof of Proposition 5.3.13. Consider $\boldsymbol{N}$ and ξ where the maximum of $I^g_{a,b}(X)$ occurs. We now apply Lemma 5.3.10 to $2m$ factors of $|f(\xi, X^{1-b\theta}, \boldsymbol{\alpha})|^{2(s-k)}$. We find

$$I^g_{a,b}(X) \leq H\left[\sum_{h=\lceil \log_2(2k)\rceil}^{H} \frac{2^m L_{h-1}}{m-k} \sum^{(m-k)2^m L_{h-1}} K^{g,h}_{a,b;m}(X) + \frac{L_H}{m}\sum^{mL_H} \mathcal{N}\right],$$

where $\mathcal{N}$ is equal to

$$\int_{[0,1[^k} |\mathfrak{F}^k_{2^{-g}X^{1-a\theta}}(\boldsymbol{N}, 2^{-g}X^{1-a\theta}, \boldsymbol{\alpha})|^2 |f(N, 2^{-H}X^{1-b\theta}, \boldsymbol{\alpha})|^{2m} |f(\xi, X^{1-b\theta}, \boldsymbol{\alpha})|^{2(s-m-k)} d\boldsymbol{\alpha}.$$

We refer to the first part as the well-spaced part and second part as the non-well-spaced part.

The well-spaced part is clearly bounded by

$$H \cdot 2^{4m-2k+2}k^{2m} \sum_{h=\lceil \log_2(2k)\rceil}^{H} \left(2^h\right)^{2(k-1)} K^{g,h}_{a,b;m}(X) \tag{5.28}$$

after inserting the bound $L_{h-1} = 2^m k^m (2^{h-1})^{k-1}$. For the non-well-spaced part, we bound $\mathcal{N}$ by Hölder's inequality. This gives the bound

$$\mathcal{N} \leq \mathcal{I}_1^{\frac{k}{s}} \mathcal{I}_2^{\frac{m}{s}} \mathcal{I}_3^{\frac{s-m-k}{s}},$$

where

$$\mathcal{I}_1 = \int_{[0,1[^k} |\mathfrak{F}^k_{2^{-g}X^{1-a\theta}}(\boldsymbol{N}, 2^{-g}X^{1-a\theta}, \boldsymbol{\alpha})|^{\frac{2s}{k}} d\boldsymbol{\alpha},$$
$$\mathcal{I}_2 = \int_{[0,1[^k} |f(N, 2^{-H}X^{1-b\theta}, \boldsymbol{\alpha})|^{2s} d\boldsymbol{\alpha},$$
$$\mathcal{I}_3 = \int_{[0,1[^k} |f(\xi, X^{1-b\theta}, \boldsymbol{\alpha})|^{2s} d\boldsymbol{\alpha}.$$

By using the inequality between the arithmetic and geometric mean on

$$|\mathfrak{F}^k_{2^{-g}X^{1-a\theta}}(\boldsymbol{N}, 2^{-g}X^{1-a\theta}, \boldsymbol{\alpha})|^{\frac{2s}{k}} \leq \frac{1}{k}\sum_{i=1}^{k} |f(N_i, 2^{-g}X^{1-a\theta}, \boldsymbol{\alpha})|^{2s}$$

and the integer translation invariance, we find that

$$\mathcal{I}_1 \leq J_{s,k}(2^{-g}X^{1-a\theta}+1),$$
$$\mathcal{I}_2 \leq J_{s,k}(2^{-H}X^{1-b\theta}+1),$$
$$\mathcal{I}_3 \leq J_{s,k}(X^{1-b\theta}+1).$$

Now, we have

$$\begin{aligned} 2^{-g}X^{1-a\theta}+1 &= 2^{-g}X^{1-a\theta}\left(1+2^{g}X^{-1+a\theta}\right) \\ &\le 2^{-g}X^{1-a\theta}\left(1+\frac{1}{4\cdot 6^6 k}\right), \end{aligned}$$

since

$$2^{g}X^{-1+a\theta}\le X^{-1+2b\theta}\cdot X^{-\theta}\le X^{-(k^2-2)b\theta}\cdot 2^{-1}\le (2k)^{-(k^2-2)}\cdot 2^{-1}\le \frac{1}{4\cdot 6^6 k}.$$

Furthermore, we have

$$\begin{aligned} 2^{-H}X^{1-b\theta}+1 &= 2^{-H}X^{1-b\theta}\left(1+2^{H}X^{-1+b\theta}\right) \\ &\le 2^{-H}X^{1-b\theta}\left(1+\frac{1}{2\cdot 6^4 k}\right) \end{aligned}$$

and

$$\begin{aligned} X^{1-b\theta}+1 &= X^{1-b\theta}\left(1+X^{-1+b\theta}\right) \\ &\le X^{1-b\theta}\left(1+\frac{1}{4\cdot 6^6 k^2}\right), \end{aligned}$$

since

$$2^{H}X^{-1+b\theta}\le X^{-1+(k+1)b\theta}\le X^{-(k^2-k-1)b\theta}\le (2k)^{-(k^2-k-1)}\le \frac{1}{2\cdot 6^4 k}$$

and

$$X^{-1+b\theta}\le X^{-(k^2-1)b\theta}\le (2k)^{-(k^2-1)}\le \frac{1}{4\cdot 6^6 k^2}.$$

Thus, we have that the non-well-spaced part is bounded by

$$\begin{aligned} &H\cdot L_H^2\cdot J_{s,k}\left(\left(1+\frac{1}{4\cdot 6^6 k}\right)2^{-g}X^{1-a\theta}\right)^{\frac{k}{s}} \\ &\quad\cdot J_{s,k}\left(\left(1+\frac{1}{2\cdot 6^4 k}\right)2^{-H}X^{1-b\theta}\right)^{\frac{m}{s}} J_{s,k}\left(\left(1+\frac{1}{4\cdot 6^6 k^2}\right)X^{1-b\theta}\right)^{\frac{s-m-k}{s}}. \end{aligned} \tag{5.29}$$

We have now that $I^g_{a,b}(X)$ is bounded by the sum of (5.28) and (5.29). By taking the maximum and normalising, we immediately see that the well-spaced part is true. In the non-well-spaced part, we are left with

$$\begin{aligned} H \cdot L_H^2 \cdot & \frac{\log\left(2\left(1+\frac{1}{4\cdot 6^6 k}\right)2^{-g}X^{1-a\theta}\right)^{\frac{k}{s}\delta}}{\log(2X)^{\frac{k}{s}\delta}} \cdot \frac{\log\left(2\left(1+\frac{1}{2\cdot 6^4 k}\right)2^{-H}X^{1-b\theta}\right)^{\frac{m}{s}\delta}}{\log(2X)^{\frac{m}{s}\delta}} \\ & \cdot \frac{\log\left(2\left(1+\frac{1}{4\cdot 6^6 k^2}\right)X^{1-b\theta}\right)^{\frac{s-m-k}{s}\delta}}{\log(2X)^{\frac{s-m-k}{s}\delta}} \cdot (2^g)^{-\frac{k}{s}(2s-\frac{1}{2}k(k+1)+\eta)} \cdot \left(2^H\right)^{-\frac{m}{s}(2s-\frac{1}{2}k(k+1)+\eta)} \\ & \cdot \left(X^{a\theta}\right)^{2k-\frac{1}{2}k(k+1)-\frac{k}{s}(2s-\frac{1}{2}k(k+1)+\eta)} \cdot \left(X^{b\theta}\right)^{2(s-k)-\frac{s-k}{s}(2s-\frac{1}{2}k(k+1)+\eta)} \\ & \cdot \left(1+\frac{1}{4\cdot 6^6 k}\right)^{\frac{k}{s}(2s-\frac{1}{2}k(k+1)+\eta)} \cdot \left(1+\frac{1}{2\cdot 6^4 k}\right)^{\frac{m}{s}(2s-\frac{1}{2}k(k+1)+\eta)} \\ & \cdot \left(1+\frac{1}{4\cdot 6^6 k^2}\right)^{\frac{s-m-k}{s}(2s-\frac{1}{2}k(k+1)+\eta)} . \end{aligned} \tag{5.30}$$

The log's are trivially bounded by 1 again and since $\eta \leq \frac{1}{2}k(k+1)$ we have

$$\begin{aligned} \left(1+\frac{1}{4\cdot 6^6 k}\right)^{\frac{k}{s}(2s-\frac{1}{2}k(k+1)+\eta)} &\leq \left(1+\frac{1}{4\cdot 6^6 k}\right)^{2k} \leq e^{\frac{1}{2\cdot 6^6}}, \\ \left(1+\frac{1}{2\cdot 6^4 k}\right)^{\frac{m}{s}(2s-\frac{1}{2}k(k+1)+\eta)} &\leq \left(1+\frac{1}{2\cdot 6^4 k}\right)^{2m} \leq e^{\frac{m}{6^4 k}} \leq 2e^{-\frac{1}{2\cdot 6^6}}, \\ \left(1+\frac{1}{4\cdot 6^6 k^2}\right)^{\frac{s-m-k}{s}(2s-\frac{1}{2}k(k+1)+\eta)} &\leq \left(1+\frac{1}{4\cdot 6^6 k^2}\right)^{2(s-k)} \leq e^{\frac{s-k}{2\cdot 6^6 k^2}} \leq k. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} \left(X^{a\theta}\right)^{2k-\frac{1}{2}k(k+1)-\frac{k}{s}(2s-\frac{1}{2}k(k+1)+\eta)} & \cdot \left(X^{b\theta}\right)^{2(s-k)-\frac{s-k}{s}(2s-\frac{1}{2}k(k+1)+\eta)} \\ &= \left(X^{\theta}\right)^{\frac{1}{2}k(k+1)\frac{s-k}{s}(b-a)} \left(X^{-\eta\theta}\right)^{\frac{k}{s}a+\frac{s-k}{s}b} . \end{aligned}$$

By inserting all of these equalities and inequalities together with $L_H = 2^m k^m (2^H)^{k-1}$ into (5.30), we find that the non-well-spaced part of the proposition is true. □

The next proposition is where we extract information as we force diagonal behaviour using Lemma 5.3.7.

Proposition 5.3.15. *Let $a, b, m \in \mathbb{N}_0$ with $b > a \geq 0$ and $6^6 k^2 \log(k) \geq s - k \geq m \geq k+1$ and $800k \geq m$. Let $\theta \in \mathbb{R}$ satisfy $1 \geq k^2 b\theta > 0$ and let $X \geq 2^{\theta^{-1}}$. Furthermore, let $g, h \in \mathbb{N}$ satisfy $X^{b\theta} \geq 2^g \geq 2k$ and $X^{kb\theta} \geq 2^h \geq 2k$. Then, we have that $[\![K^{g,h}_{a,b;m}(X)]\!]$ is bounded by*

$$C' \cdot (2^g)^{-k+\frac{1}{2}k(k-1)} \left(2^h\right)^{-(2s-\frac{1}{2}k(k+1)+\eta)\left(\frac{m}{s}-\frac{k^2}{s(s-k)}\right)} \cdot [\![I^h_{b,kb}(X)]\!]^{\frac{k}{s-k}} \cdot X^{-\frac{s-2k}{s-k}b\theta\cdot\eta},$$

where

$$C' = 2^{\frac{1}{2}k(k+1)+5} e^{\frac{1}{4}k(3k-2)} k^{-\frac{1}{2}k(k-2)+1} \cdot (s-k+k^2)^k.$$

Proof. Consider an $\boldsymbol{N}, \boldsymbol{N'}, N', \xi$, where the maximum of $K^{g,h}_{a,b;m}(X)$ occurs, and its corresponding Diophantine equation:

$$\sum_{i=1}^{k}(x_i^j - y_i^j) = \sum_{i=1}^{k}(w_i^j - z_i^j) + \sum_{i=1}^{m-k}(u_i^j - v_i^j) + \sum_{i=1}^{s-m-k}(p_i^j - q_i^j), \quad j = 1, \dots, k, \tag{5.31}$$

where $\boldsymbol{x}, \boldsymbol{y} \in \mathfrak{B}^k(\boldsymbol{N}, 2^{-g}X^{1-a\theta})$, $\boldsymbol{w}, \boldsymbol{z} \in \mathfrak{B}^k(\boldsymbol{N'}, 2^{-h}X^{1-b\theta})$, $\boldsymbol{u}, \boldsymbol{v} \in \mathfrak{B}^{m-k}(N', 2^{-h}X^{1-b\theta})$ and $\boldsymbol{w}, \boldsymbol{z}, \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{p}, \boldsymbol{q} \in \mathfrak{B}^{s-m-k}(\xi, X^{1-b\theta})$. The right-hand side is contained in

$$\left]-2(s-k)\,(0.5001)^j\, X^{(1-b\theta)j}, 2(s-k)\,(0.5001)^j\, X^{(1-b\theta)j}\right[\tag{5.32}$$

as

$$\begin{aligned}
\xi + \frac{1}{2}X^{1-b\theta} &\le \frac{1}{2}X^{1-b\theta}\left(1 + X^{-1+b\theta}\right) \\
&\le \frac{1}{2}X^{1-b\theta}\left(1 + X^{-(k^2-1)b\theta}\right) \\
&\le \frac{1}{2}X^{1-b\theta}\left(1 + (2k)^{-(k^2-1)}\right) \\
&\le 0.5001 \cdot X^{1-b\theta}.
\end{aligned}$$

The interval (5.32) we split up into intervals V_j of size at most

$$(s-k)X^{1-kb\theta} \cdot X^{j-1}.$$

We have at most

$$\begin{aligned}
\prod_{j=1}^{k}\left(4\,(0.5001)^j\, X^{(k-j)b\theta} + 1\right) &\le \prod_{j=1}^{\infty}\left(1 + 4\,(0.5001)^j\right) \cdot X^{\frac{1}{2}k(k-1)b\theta} \\
&\le 2^4 \cdot X^{\frac{1}{2}k(k-1)b\theta} = Z', \text{ say,}
\end{aligned}$$

choices for $\boldsymbol{V} = (V_j)_j$ as

$$\prod_{j=1}^{\infty}\left(1 + 4\,(0.5001)^j\right) \le \prod_{j=1}^{10}\left(1 + 4\,(0.5001)^j\right) \cdot \exp\left(4\sum_{j=11}^{\infty}(0.5001)^j\right) < 14.27 \cdot 1.004 < 2^4.$$

Furthermore, we split up the box $\mathfrak{B}^k(\boldsymbol{N}, 2^{-g}X^{1-a\theta})$ for the $\boldsymbol{y}$'s into sub-boxes of the shape $\mathfrak{B}^k(\boldsymbol{N''}, X^{1-kb\theta})$. We have at most

$$\begin{aligned}
\left(\frac{X^{(kb-a)\theta}}{2^g} + 1\right)^k &= \left(1 + 2^g X^{-(kb-a)\theta}\right)^k \frac{X^{(kb-a)k\theta}}{2^{gk}} \\
&\le e^{\frac{1}{4}} \frac{X^{(kb-a)k\theta}}{2^{gk}} = Z''_g, \text{ say,}
\end{aligned}$$

of these, since

$$2^g X^{-(kb-a)\theta} \leq X^{-((k-1)b-a)\theta} \leq X^{-(k-2)b\theta} X^{-\theta} \leq (2k)^{-(k-2)} \cdot 2^{-1}$$

and

$$\left(1 + (2k)^{-(k-2)} \cdot 2^{-1}\right)^k \leq e^{\frac{1}{4(2k)^{k-3}}} \leq e^{\frac{1}{4}}.$$

Let $\mathcal{S}(\mathfrak{B}^k(\boldsymbol{N''}, X^{1-kb\theta}), \boldsymbol{V})$ denote the number of solutions $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{w}, \boldsymbol{z}, \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{p}, \boldsymbol{q})$ of (5.31) with the additional restriction that

$$\sum_{i=1}^{k} (x_i^j - y_i^j) \in V_j \quad (j = 1, \ldots, k)$$

and $\boldsymbol{y} \in \mathfrak{B}^k(\boldsymbol{N''}, X^{1-kb\theta})$, so that the total number of solutions to (5.31) is bounded by

$$\sum^{Z'} \sum^{Z''_g} \mathcal{S}(\mathfrak{B}^k(\boldsymbol{N''}, X^{1-kb\theta}), \boldsymbol{V}). \tag{5.33}$$

Two solutions $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{w}, \boldsymbol{z}, \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{p}, \boldsymbol{q}), (\boldsymbol{x'}, \boldsymbol{y'}, \boldsymbol{w'}, \boldsymbol{z'}, \boldsymbol{u'}, \boldsymbol{v'}, \boldsymbol{p'}, \boldsymbol{q'})$ of $\mathcal{S}(\mathfrak{B}^k(\boldsymbol{N'}, X^{1-kb\theta}), \boldsymbol{V})$ satisfy the inequality

$$\begin{aligned} \left| \sum_{i=1}^{k} x_i^j - \sum_{i=1}^{k} x_i'^j \right| &\leq \left| \sum_{i=1}^{k} (x_i^j - y_i^j) - \sum_{i=1}^{k} (x_i'^j - y_i'^j) \right| + \left| \sum_{i=1}^{k} y_i^j - \sum_{i=1}^{k} y_i'^j \right| \\ &\leq (s-k) X^{1-kb\theta} \cdot X^{j-1} + jk X^{1-kb\theta} \cdot X^{j-1} \\ &\leq (s-k+k^2) X^{1-kb\theta} \cdot X^{j-1}. \end{aligned} \tag{5.34}$$

Thus, we are able to apply Lemma 5.3.7 to bound $\mathcal{S}(\mathfrak{B}^k(\boldsymbol{N''}, X^{1-kb\theta}), \boldsymbol{V})$. We are able to apply it with $S = (s-k+k^2)X^{1-kb\theta}$, $R = 2^{-g}X^{1-a\theta}$, $\boldsymbol{U}$ the interval in (5.34), and $W(\boldsymbol{x})$ being the number of solutions $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{w}, \boldsymbol{z}, \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{p}, \boldsymbol{q})$ counted by $\mathcal{S}(\mathfrak{B}^k(\boldsymbol{N''}, X^{1-kb\theta}), \boldsymbol{V})$. We get that (5.33) is bounded by

$$\sum^{Z'} \sum^{Z''_g} 2^{\frac{1}{2}k(k+1)} e^{\frac{1}{4}(3k+1)(k-1)} k^{-\frac{1}{2}k(k-2)} \cdot \left(2^g X^{a\theta}\right)^{\frac{1}{2}k(k-1)} \cdot \mathcal{Z}_W(\mathfrak{B}^k(\boldsymbol{N'''}, \boldsymbol{S'}), \boldsymbol{U}), \tag{5.35}$$

with $1 \leq \boldsymbol{S'} \leq (s-k+k^2)X^{1-kb\theta}$. Now, $\mathcal{Z}_W(\mathfrak{B}^k(\boldsymbol{N'''}, \boldsymbol{S'}), \boldsymbol{U})$ is just counting the number of solutions of (5.31) with some further restrictions. The two we care about are $\boldsymbol{x} \in \mathfrak{B}^k(\boldsymbol{N'''}, \boldsymbol{S'})$ and $\boldsymbol{y} \in \mathfrak{B}^k(\boldsymbol{N''}, X^{1-kb\theta})$. Therefore, we have

$$\begin{aligned} &\mathcal{Z}_W(\mathfrak{B}^k(\boldsymbol{N'''}, \boldsymbol{S'}), \boldsymbol{U}) \\ &\qquad \leq \int_{[0,1[^k} \mathfrak{F}^k_{2^{-g}X^{1-a\theta}}(\boldsymbol{N'''}, \boldsymbol{S'}, \boldsymbol{\alpha}) \mathfrak{F}^k_{2^{-g}X^{1-a\theta}}(\boldsymbol{N''}, X^{1-kb\theta}, -\boldsymbol{\alpha}) \cdot \mathfrak{f}^\star d\boldsymbol{\alpha}, \end{aligned} \tag{5.36}$$

where

$$\mathfrak{f}^\star = |\mathfrak{F}^k_{2^{-h}X^{1-b\theta}}(\boldsymbol{N'}, 2^{-h}X^{1-b\theta}, \boldsymbol{\alpha})|^2 |f(N', 2^{-h}X^{1-b\theta}, \boldsymbol{\alpha})|^{2(m-k)} |f(\xi, X^{1-b\theta}, \boldsymbol{\alpha})|^{2(s-m-k)}.$$

We split up $\mathfrak{B}^k(\boldsymbol{N'''}, \boldsymbol{S'})$ further into $(s-k+k^2)^k$ sub-boxes of size at most $X^{1-kb\theta}$, which we may assume to have exactly size $X^{1-kb\theta}$. Thus, the integral in (5.36) is further bounded by

$$\sum^{(s-k+k^2)^k} \int_{[0,1[^k} \mathfrak{F}^k_{2^{-g}X^{1-a\theta}}(\boldsymbol{N''''}, X^{1-kb\theta}, \boldsymbol{\alpha}) \mathfrak{F}^k_{2^{-g}X^{1-a\theta}}(\boldsymbol{N''}, X^{1-kb\theta}, -\boldsymbol{\alpha}) \cdot \mathfrak{f}^\star d\boldsymbol{\alpha}.$$

By using Hölder's inequality, we further find

$$\mathcal{Z}_W(\mathfrak{B}^k(\boldsymbol{N'''}, \boldsymbol{S'}), \boldsymbol{U}) \leq \sum^{(s-k+k^2)^k} \mathcal{I}_1^{\frac{k}{2(s-k)}} \mathcal{I}_2^{\frac{k}{2(s-k)}} \mathcal{I}_3^{\frac{(s-2k)k}{s(s-k)}} \mathcal{I}_4^{\frac{m-k}{s}} \mathcal{I}_5^{\frac{s-m-k}{s}}, \tag{5.37}$$

where

$$\mathcal{I}_1 = \int_{[0,1[^k} |\mathfrak{F}^k_{2^{-h}X^{1-b\theta}}(\boldsymbol{N'}, 2^{-h}X^{1-b\theta}, \boldsymbol{\alpha})|^2 |\mathfrak{F}^k_{2^{-g}X^{1-a\theta}}(\boldsymbol{N''''}, X^{1-kb\theta}, \boldsymbol{\alpha})|^{\frac{2(s-k)}{k}} d\boldsymbol{\alpha},$$
$$\mathcal{I}_2 = \int_{[0,1[^k} |\mathfrak{F}^k_{2^{-h}X^{1-b\theta}}(\boldsymbol{N'}, 2^{-h}X^{1-b\theta}, \boldsymbol{\alpha})|^2 |\mathfrak{F}^k_{2^{-g}X^{1-a\theta}}(\boldsymbol{N''}, X^{1-kb\theta}, \boldsymbol{\alpha})|^{\frac{2(s-k)}{k}} d\boldsymbol{\alpha},$$
$$\mathcal{I}_3 = \int_{[0,1[^k} |\mathfrak{F}^k_{2^{-h}X^{1-b\theta}}(\boldsymbol{N'}, 2^{-h}X^{1-b\theta}, \boldsymbol{\alpha})|^{\frac{2s}{k}} d\boldsymbol{\alpha},$$
$$\mathcal{I}_4 = \int_{[0,1[^k} |f(N', 2^{-h}X^{1-b\theta}, \boldsymbol{\alpha})|^{2s} d\boldsymbol{\alpha},$$
$$\mathcal{I}_5 = \int_{[0,1[^k} |f(\xi, X^{1-b\theta}, \boldsymbol{\alpha})|^{2s} d\boldsymbol{\alpha}.$$

By using the inequality between the arithmetic and geometric mean on

$$|\mathfrak{F}^k_{2^{-g}X^{1-a\theta}}(\boldsymbol{N''}, X^{1-kb\theta}, \boldsymbol{\alpha})|^{\frac{2(s-k)}{k}} \leq \frac{1}{k} \sum_{i=1}^{k} |f(N_i'', X^{1-kb\theta}, \boldsymbol{\alpha})|^{2(s-k)}$$

and the integer translation invariance, we find that

$$\mathcal{I}_1 \leq I^h_{b,bk}(X).$$

Analogously, also

$$\mathcal{I}_2 \leq I^h_{b,bk}(X)$$

holds. Using the inequality between the arithmetic and geometric mean in a similar fashion, we find by the integer translation invariance that

$$\mathcal{I}_3, \mathcal{I}_4 \leq J_{s,k}\left(2^{-h}X^{1-b\theta} + 1\right).$$

Again, by the integer translation invariance, we find that

$$\mathcal{I}_5 \le J_{s,k}\left(X^{1-b\theta}+1\right).$$

We have

$$\begin{aligned}2^{-h}X^{1-b\theta}+1 &= 2^{-h}X^{1-b\theta}\left(1+2^{h}X^{-1+b\theta}\right)\\ &\le 2^{-h}X^{1-b\theta}\left(1+\frac{1}{2\cdot 6^4 k}\right),\end{aligned}$$

since

$$2^h X^{-1+b\theta} \le X^{-1+(k+1)b\theta} \le X^{-(k^2-k-1)b\theta} \le (2k)^{-(k^2-k-1)} \le \frac{1}{2\cdot 6^4 k},$$

and

$$\begin{aligned}X^{1-b\theta}+1 &= X^{1-b\theta}\left(1+X^{-1+b\theta}\right)\\ &\le X^{1-b\theta}\left(1+\frac{1}{4\cdot 6^6 k^2}\right),\end{aligned}$$

since

$$X^{-1+b\theta} \le X^{-(k^2-1)b\theta} \le (2k)^{-(k^2-1)} \le \frac{1}{4\cdot 6^6 k^2}.$$

By inserting the above analysis into (5.37) and further into (5.35), we conclude that $K^{g,h}_{a,b;m}(X)$ is bounded by

$$\begin{aligned}&2^4 X^{\frac{1}{2}k(k-1)b\theta} e^{\frac{1}{4}} 2^{-gk} X^{(kb-a)k\theta} 2^{\frac{1}{2}k(k+1)} e^{\frac{1}{4}(3k+1)(k-1)} k^{-\frac{1}{2}k(k-2)} (s-k+k^2)^k\\ &\quad\cdot \left(2^g X^{a\theta}\right)^{\frac{1}{2}k(k-1)} I^h_{b,bk}(X)^{\frac{k}{s-k}} J_{s,k}\left(\left(1+\frac{1}{2\cdot 6^4 k}\right)2^{-h}X^{1-b\theta}\right)^{\frac{m}{s}-\frac{k^2}{s(s-k)}}\\ &\qquad\cdot J_{s,k}\left(\left(1+\frac{1}{4\cdot 6^6 k^2}\right)X^{1-b\theta}\right)^{\frac{s-m-k}{s}}.\end{aligned}$$

Let us apply the normalisations and analyse each parameter separately. The dependence on X is going to be

$$\begin{aligned}&\frac{\log\left(2\left(1+\frac{1}{2\cdot 6^4 k}\right)2^{-h}X^{1-b\theta}\right)^{\delta\left(\frac{m}{s}-\frac{k^2}{s(s-k)}\right)}}{\log(2X)^{\delta\left(\frac{m}{s}-\frac{k^2}{s(s-k)}\right)}}\cdot\frac{\log\left(2\left(1+\frac{1}{4\cdot 6^6 k^2}\right)X^{1-b\theta}\right)^{\delta\frac{s-m-k}{s}}}{\log(2X)^{\delta\frac{s-m-k}{s}}}\\ &\cdot\left(X^{a\theta}\right)^{2k-\frac{1}{2}k(k+1)-k+\frac{1}{2}k(k-1)}\left(X^{b\theta}\right)^{2(s-k)+\frac{1}{2}k(k-1)+k^2-\frac{k}{s-k}\left(2k-\frac{1}{2}k(k+1)+2(s-k)k\right)}\\ &\cdot\left(X^{b\theta}\right)^{-\frac{s-2k}{s-k}\left(2s-\frac{1}{2}k(k+1)+\eta\right)}.\end{aligned}$$

The fraction with $\log$'s are bounded by 1 again. The exponent of $X^{a\theta}$ is 0 and the exponent of $X^{b\theta}$ reduces to $-\frac{s-2k}{s-k}\eta$ after a short computation. The dependence on h is

$$\left(2^h\right)^{-\left(2s-\frac{1}{2}k(k+1)+\eta\right)\left(\frac{m}{s}-\frac{k^2}{s(s-k)}\right)}.$$

The dependence on g is
$$(2^g)^{-k+\frac{1}{2}k(k-1)}.$$

And finally, the constant is

$$2^4 \cdot e^{\frac{1}{4}} \cdot 2^{\frac{1}{2}k(k+1)} e^{\frac{1}{4}(3k+1)(k-1)} \cdot k^{-\frac{1}{2}k(k-2)} \cdot (s-k+k^2)^k$$
$$\cdot \left(1+\frac{1}{2\cdot 6^4 k}\right)^{\left(2s-\frac{1}{2}k(k+1)+\eta\right)\left(\frac{m}{s}-\frac{k^2}{s(s-k)}\right)} \cdot \left(1+\frac{1}{4\cdot 6^6 k^2}\right)^{\left(2s-\frac{1}{2}k(k+1)+\eta\right)\frac{s-m-k}{s}}.$$

Since $\eta \leq \frac{1}{2}k(k+1)$, we have

$$\left(1+\frac{1}{2\cdot 6^4 k}\right)^{\left(2s-\frac{1}{2}k(k+1)+\eta\right)\left(\frac{m}{s}-\frac{k^2}{s(s-k)}\right)} \leq e^{\frac{2s}{2\cdot 6^4 k}\frac{m}{s}} \leq e^{\frac{m}{6^4 k}} \leq 2$$

and

$$\left(1+\frac{1}{4\cdot 6^6 k^2}\right)^{\left(2s-\frac{1}{2}k(k+1)+\eta\right)\frac{s-m-k}{s}} \leq e^{\frac{2s}{4\cdot 6^6 k^2}\frac{s-k}{s}} \leq e^{\frac{s-k}{2\cdot 6^6 k^2}} \leq k.$$

Thus, we find that the constant is bounded by

$$2^{\frac{1}{2}k(k+1)+5} e^{\frac{1}{4}k(3k-2)} k^{-\frac{1}{2}k(k-2)+1} \cdot (s-k+k^2)^k.$$

□

This final proposition is essentially the same as the previous one, the difference being that the iteration comes to a halt after this step.

Proposition 5.3.16. *Let $a, b \in \mathbb{N}_0$ with $b > a$. Let $\theta \in \mathbb{R}$ satisfy $1 \geq kb\theta > 0$ and let $X \geq 2^{\theta^{-1}}$. Furthermore, let $g \in \mathbb{N}$ satisfy $X^{b\theta} \geq 2^g \geq 2k$ and $2k^2 \log(k) \geq s-k$. Then, we have:*

$$[\![I^g_{a,b}(X)]\!] \leq C' \cdot (2^g)^{-k+\frac{1}{2}k(k-1)} X^{\frac{k^2(k^2-1)}{2s}b\theta} X^{-\eta\frac{s+k^2-k}{s}b\theta},$$

where

$$C' = 2^{\frac{1}{2}k(k+5)+4} \cdot e^{\frac{1}{4}k(3k-2)} \cdot k^{-\frac{1}{2}k(k-2)+1} \cdot (s-k+k^2)^k.$$

Proof. Consider an $\boldsymbol{N}, \xi$, where the maximum of $I^g_{a,b}(X)$ occurs, and its corresponding Diophantine equation:

$$\sum_{i=1}^{k}(x_i^j - y_i^j) = \sum_{i=1}^{s-k}(p_i^j - q_i^j), \quad j = 1, \ldots, k, \tag{5.38}$$

where $\boldsymbol{x}, \boldsymbol{y} \in \mathfrak{B}^k(\boldsymbol{N}, 2^{-g}X^{1-a\theta})$ and $\boldsymbol{p}, \boldsymbol{q} \in \mathfrak{B}^{s-k}(\xi, X^{1-b\theta})$.

From here, we proceed as in the previous proposition, but in this case we need to adjust our definition of $\mathcal{S}(\mathfrak{B}^k(\boldsymbol{N''}, X^{1-kb\theta}), \boldsymbol{V})$. Let $\mathcal{S}(\mathfrak{B}^k(\boldsymbol{N''}, X^{1-kb\theta}), \boldsymbol{V})$ denote the number of solutions $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{p}, \boldsymbol{q})$ of (5.38) with the additional restriction that

$$\sum_{i=1}^{k}(x_i^j - y_i^j) \in V_j \quad (j = 1, \ldots, k)$$

and $\boldsymbol{y} \in \mathfrak{B}^k(\boldsymbol{N''}, X^{1-kb\theta})$, so that the total number of solutions to (5.38) is bounded by

$$\sum^{Z'}\sum^{Z''_g} \mathcal{S}(\mathfrak{B}^k(\boldsymbol{N''}, X^{1-kb\theta}), \boldsymbol{V}). \tag{5.39}$$

Consider now two solutions $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{p}, \boldsymbol{q}), (\boldsymbol{x'}, \boldsymbol{y'}, \boldsymbol{p'}, \boldsymbol{q'})$ of $\mathcal{S}(\mathfrak{B}^k(\boldsymbol{N'}, X^{1-kb\theta}), \boldsymbol{V})$. In this case, we have

$$\begin{aligned} \left|\sum_{i=1}^{k} x_i^j - \sum_{i=1}^{k} x_i'^j\right| &\le \left|\sum_{i=1}^{k}(x_i^j - y_i^j) - \sum_{i=1}^{k}(x_i'^j - y_i'^j)\right| + \left|\sum_{i=1}^{k} y_i^j - \sum_{i=1}^{k} y_i'^j\right| \\ &\le (s-k)X^{1-kb\theta} \cdot X^{j-1} + jkX^{1-kb\theta} \cdot X^{j-1} \\ &\le (s-k+k^2)X^{1-kb\theta} \cdot X^{j-1}. \end{aligned} \tag{5.40}$$

Thus, we are again able to apply Lemma 5.3.7 to bound $\mathcal{S}(\mathfrak{B}^k(\boldsymbol{N''}, X^{1-kb\theta}), \boldsymbol{V})$. We use the lemma with the weight function $W(\boldsymbol{x})$ being the number of solutions $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{p}, \boldsymbol{q})$ counted by $\mathcal{S}(\mathfrak{B}^k(\boldsymbol{N''}, X^{1-kb\theta}), \boldsymbol{V})$ and $\boldsymbol{U}$ being the interval in (5.40). We arrive at the conclusion that (5.39) is bounded by

$$\sum^{Z'}\sum^{Z''_g} 2^{\frac{1}{2}k(k+1)} e^{\frac{1}{4}(3k+1)(k-1)} k^{-\frac{1}{2}k(k-2)} \cdot \left(2^g X^{a\theta}\right)^{\frac{1}{2}k(k-1)} \cdot \mathcal{Z}_W(\mathfrak{B}^k(\boldsymbol{N'''}, \boldsymbol{S'}), \boldsymbol{U}), \tag{5.41}$$

with $1 \le \boldsymbol{S'} \le (s-k+k^2)X^{1-kb\theta}$. Now, $\mathcal{Z}_W(\mathfrak{B}^k(\boldsymbol{N'''}, \boldsymbol{S'}), \boldsymbol{U})$ is just counting the number of solutions of (5.38) with some further restrictions. The two we care about are $\boldsymbol{x} \in \mathfrak{B}^k(\boldsymbol{N'''}, \boldsymbol{S'})$ and $\boldsymbol{y} \in \mathfrak{B}^k(\boldsymbol{N''}, X^{1-kb\theta})$. Thus, we arrive at

$$\begin{aligned} &\mathcal{Z}_W(\mathfrak{B}^k(\boldsymbol{N'''}, \boldsymbol{S'}), \boldsymbol{U}) \\ &\le \int_{[0,1[^k} \mathfrak{F}^k_{2^{-g}X^{1-a\theta}}(\boldsymbol{N'''}, \boldsymbol{S'}, \boldsymbol{\alpha}) \mathfrak{F}^k_{2^{-g}X^{1-a\theta}}(\boldsymbol{N''}, X^{1-kb\theta}, -\boldsymbol{\alpha}) |f(\xi, X^{1-b\theta}, \boldsymbol{\alpha})|^{2(s-k)} d\boldsymbol{\alpha}. \end{aligned} \tag{5.42}$$

We split up $\mathfrak{B}^k(\boldsymbol{N'''}, \boldsymbol{S'})$ further into $(s-k+k^2)^k$ sub-boxes of size at most $X^{1-kb\theta}$, which we may assume to have exactly size $X^{1-kb\theta}$. Thus, the integral in (5.42) is further bounded by

$$\begin{aligned} \sum^{(s-k+k^2)^k} \int_{[0,1[^k} \mathfrak{F}^k_{2^{-g}X^{1-a\theta}}(\boldsymbol{N''''}, X^{1-kb\theta}, \boldsymbol{\alpha}) \mathfrak{F}^k_{2^{-g}X^{1-a\theta}}(\boldsymbol{N''}, X^{1-kb\theta}, -\boldsymbol{\alpha}) \\ \cdot |f(\xi, X^{1-b\theta}, \boldsymbol{\alpha})|^{2(s-k)} d\boldsymbol{\alpha}. \end{aligned}$$

We further find by using Hölder's inequality, that

$$\mathcal{Z}_W(\mathfrak{B}^k(\boldsymbol{N}''',\boldsymbol{S}'),\boldsymbol{U}) \leq \sum^{(s-k+k^2)^k} \mathcal{I}_1^{\frac{k}{2s}}\mathcal{I}_2^{\frac{k}{2s}}\mathcal{I}_3^{\frac{s-k}{s}}, \tag{5.43}$$

where

$$\begin{aligned}\mathcal{I}_1 &= \int_{[0,1[^k} |\mathfrak{F}^k_{2^{-g}X^{1-a\theta}}(\boldsymbol{N}'', X^{1-kb\theta},\boldsymbol{\alpha})|^{\frac{2s}{k}}d\boldsymbol{\alpha},\\ \mathcal{I}_2 &= \int_{[0,1[^k} |\mathfrak{F}^k_{2^{-g}X^{1-a\theta}}(\boldsymbol{N}, X^{1-kb\theta},\boldsymbol{\alpha})|^{\frac{2s}{k}}d\boldsymbol{\alpha},\\ \mathcal{I}_3 &= \int_{[0,1[^k} |f(\xi, X^{1-b\theta},\boldsymbol{\alpha})|^{2s}d\boldsymbol{\alpha}.\end{aligned}$$

By using the inequality between the arithmetic and geometric mean on

$$|\mathfrak{F}^k_{2^{-g}X^{1-a\theta}}(\boldsymbol{N}'', X^{1-kb\theta},\boldsymbol{\alpha})|^{\frac{2s}{k}} \leq \frac{1}{k}\sum_{i=1}^{k}|f(N_i'', X^{1-kb\theta},\boldsymbol{\alpha})|^{2s}$$

and the integer translation invariance, we find that

$$\mathcal{I}_1 \leq J_{s,k}(X^{1-kb\theta}+1) \leq J_{s,k}(2X^{1-kb\theta}).$$

Analogously, also

$$\mathcal{I}_2 \leq J_{s,k}(X^{1-kb\theta}+1) \leq J_{s,k}(2X^{1-kb\theta})$$

holds. And finally, by the integer translation invariance, we find

$$\mathcal{I}_3 \leq J_{s,k}\left(X^{1-b\theta}+1\right).$$

We have

$$\begin{aligned}X^{1-b\theta}+1 &= X^{1-b\theta}\left(1+X^{-1+b\theta}\right)\\ &\leq X^{1-b\theta}\left(1+\frac{1}{4k^2}\right),\end{aligned}$$

since

$$X^{-1+b\theta} \leq X^{-(k-1)b\theta} \leq (2k)^{-(k-1)} \leq \frac{1}{4k^2}.$$

By inserting the above analysis into (5.43) and further (5.41), we conclude that $I^g_{a,b}(X)$ is bounded by

$$\begin{aligned}2^4X^{\frac{1}{2}k(k-1)b\theta}e^{\frac{1}{4}}2^{-gk}X^{(kb-a)k\theta}2^{\frac{1}{2}k(k+1)}e^{\frac{1}{4}(3k+1)(k-1)}k^{-\frac{1}{2}k(k-2)}\cdot\left(2^gX^{a\theta}\right)^{\frac{1}{2}k(k-1)}\\ \cdot(s-k+k^2)^k\cdot J_{s,k}(2X^{1-kb\theta})^{\frac{k}{s}}J_{s,k}\left(\left(1+\frac{1}{4k^2}\right)X^{1-b\theta}\right)^{\frac{s-k}{s}}.\end{aligned}$$

Let us apply the normalisations and analyse each parameter separately. The dependence on X is going to be

$$\frac{\log\left(4X^{1-kb\theta}\right)^{\delta\frac{k}{s}}}{\log(2X)^{\delta\frac{k}{s}}}\cdot\frac{\log\left(2\left(1+\frac{1}{4k^2}\right)X^{1-b\theta}\right)^{\delta\frac{s-k}{s}}}{\log(2X)^{\delta\frac{s-k}{s}}}\cdot\left(X^{a\theta}\right)^{2k-\frac{1}{2}k(k+1)-k+\frac{1}{2}k(k-1)}$$
$$\cdot\left(X^{b\theta}\right)^{2(s-k)+\frac{1}{2}k(k-1)+k^2-\frac{k^2}{s}\left(2s-\frac{1}{2}k(k+1)+\eta\right)-\frac{s-k}{s}\left(2s-\frac{1}{2}k(k+1)+\eta\right)}.$$

The fraction with $\log$'s are bounded by 1 again. The exponent of $X^{a\theta}$ is 0 and the exponent of $X^{b\theta}$ reduces to $\frac{k^2(k^2-1)}{2s}-\frac{s-k+k^2}{s}\eta$ after a short computation. The dependence on g is $(2^g)^{-k+\frac{1}{2}k(k-1)}$. And finally, the constant is

$$2^4\cdot e^{\frac{1}{4}}\cdot 2^{\frac{1}{2}k(k+1)}e^{\frac{1}{4}(3k+1)(k-1)}\cdot k^{-\frac{1}{2}k(k-2)}\cdot(s-k+k^2)^k$$
$$\cdot 2^{\left(2s-\frac{1}{2}k(k+1)+\eta\right)\frac{k}{s}}\cdot\left(1+\frac{1}{4k^2}\right)^{\left(2s-\frac{1}{2}k(k+1)+\eta\right)\frac{s-k}{s}}.$$

Since $\eta\leq\frac{1}{2}k(k+1)$, we have

$$2^{\left(2s-\frac{1}{2}k(k+1)+\eta\right)\frac{k}{s}}\leq 2^{2k}$$

and

$$\left(1+\frac{1}{4k^2}\right)^{\left(2s-\frac{1}{2}k(k+1)+\eta\right)\frac{s-k}{s}}\leq e^{\frac{2s}{4k^2}\frac{s-k}{s}}\leq e^{\frac{s-k}{2k^2}}\leq k.$$

Therefore, we see that the constant is bounded by

$$2^4\cdot 2^{\frac{1}{2}k(k+1)}e^{\frac{1}{4}k(3k-2)}k^{-\frac{1}{2}k(k-2)}\cdot(s-k+k^2)^k\cdot 2^{2k}\cdot k.$$

□

5.3.5 *Iterative Process*

In this section, we iterate through the Propositions 5.3.13 and 5.3.15 as often as we can. This was already outlined in Section 5.3.2 and we recommend the reader to have a second look at it before advancing, since the argument to follow is essentially the same with the exception that there are more parameters to be analysed and chosen.

Let us recall some of our notation of the outline. Let $D\geq 1$ be an integer and set $\theta=k^{-(D+1)}$. Let $(a_0,b_0),(a_1,b_1),(a_2,b_2),\ldots,(a_D,b_D)$ denote the sequence

$$(0,1),(1,k),(k,k^2),\ldots,(k^{D-1},k^D).$$

Furthermore, we assume $X \geq 2^{k^{D+1}}$ and $2\log(k) \geq \lambda = \frac{s-k}{k^2} \geq 1$. We now fix a choice of parameters, which we will justify later on. Set

$$G_n = \lfloor k^n \theta \log_2(X) \rfloor, \quad \text{for } n = 0, \ldots, D$$

and

$$m_n = \begin{cases} \left\lfloor \frac{1}{4}k(k+1) + \frac{4}{3}k - \frac{1}{2} \right\rfloor, & \text{if } n = 0, \\ \left\lfloor \frac{5}{3}k \right\rfloor, & \text{if } n = 1, \ldots, D. \end{cases} \tag{5.44}$$

We remark here, that the choice of G_n will ensure that the conditions

$$X^{kb_{n-1}\theta} = X^{b_n\theta} \geq 2^{G_n} \geq 2^{g_n} \geq 2k \tag{5.45}$$

of the Propositions 5.3.12, 5.3.13, 5.3.15 and 5.3.16 are satisfied, where the last inequality comes from the restriction of our well-spaced parameter g_n in Lemma 5.3.10. We would also like to highlight the inequalities

$$\frac{1}{4}k(k+1) + \frac{4}{3}k - \frac{1}{2} \geq m_0 \geq \frac{1}{4}k(k+1) + \frac{4}{3}k - \frac{4}{3}$$

and

$$\frac{5}{3}k \geq m_n \geq \frac{5}{3}k - \frac{2}{3}, \quad \forall n = 1, \ldots, D,$$

which will be frequently used. The conditions of Proposition 5.3.12 are now clearly met. Thus, we get

$$\begin{aligned} [\![J_{s,k}(X)]\!] \leq & C' \cdot G_0 \sum_{g_0 = \lceil \log_2(2k) \rceil}^{G_0} \left(2^{g_0}\right)^{2(k-1)-2(m_0-k)} [\![I_{0,1}^{g_0}(X)]\!] \\ & + C'' \cdot G_0 \left(2^{G_0}\right)^{2(k-1) - \frac{m_0}{s}\left(2s - \frac{1}{2}k(k+1) + \eta\right)}, \end{aligned}$$

where

$$\begin{aligned} C' &= 2^{6m_0 - 4k + 2} k^{2m_0} \cdot \left(1 + \frac{1}{X^\theta}\right)^{2(s-m_0)}, \\ C'' &= 2^{2m_0+1} k^{2m_0}. \end{aligned}$$

By using the inequalities on m_0 and $G_0 \leq k^{-(D+1)} \log_2(2X)$, we find

$$[\![J_{s,k}(X)]\!] \leq \log_2(2X) \left(C_0 \sum_{g_0 = \lceil \log_2(2k) \rceil}^{G_0} \left(2^{g_0}\right)^{2(k-1)-2(m_0-k)} [\![I_{0,1}^{g_0}(X)]\!] + E_0 \right),$$

where

$$C_0 = 2^{\frac{3}{2}k^2 + \frac{11}{2}k - 1} k^{\frac{1}{2}k^2 + \frac{19}{6}k - 2 - D} \left(1 + \frac{1}{X^\theta}\right)^{2(s-m_0)}$$

and

$$E_0 = 2^{\frac{1}{2}k^2+\frac{19}{6}k} k^{\frac{1}{2}k^2+\frac{19}{6}k-2-D} \left(2^{G_0}\right)^{2(k-1)-\frac{m_0}{s}(2s-\frac{1}{2}k(k+1)+\eta)}.$$

We have

$$\frac{2s-\frac{1}{2}k(k+1)+\eta}{s} \geq \frac{3}{2} + \frac{\eta}{k(k+1)} \Leftrightarrow \left(\frac{1}{2}k(k+1)-\eta\right)\left(\frac{s}{k(k+1)}-1\right) \geq 0$$

and therefore we further find

$$\begin{aligned} E_0 &\leq 2^{\frac{1}{2}k^2+\frac{19}{6}k} k^{\frac{1}{2}k^2+\frac{19}{6}k-2-D} \left(2^{G_0}\right)^{2(k-1)-\frac{3}{2}m_0-\frac{m_0}{k(k+1)}\eta} \\ &\leq 2^{\frac{1}{2}k^2+\frac{19}{6}k} k^{\frac{1}{2}k^2+\frac{19}{6}k-2-D} \left(2^{G_0}\right)^{-\frac{3}{8}k(k+1)-\frac{1}{4}\eta} \\ &\leq 2^{\frac{1}{2}k^2+\frac{19}{6}k} k^{\frac{1}{2}k^2+\frac{19}{6}k-2-D} \left(\frac{X^\theta}{2}\right)^{-\frac{3}{8}k(k+1)-\frac{1}{4}\eta} \\ &\leq 2^{k^2+\frac{11}{3}k} k^{\frac{1}{2}k^2+\frac{19}{6}k-2-D} X^{-\eta\theta}. \end{aligned}$$

In conclusion, we have

$$[\![J_{s,k}(X)]\!] \leq \log_2(2X)\cdot\Psi_0,$$

where

$$\Psi_0 = \mathcal{C}_0 \sum_{g_0=\lceil\log_2(2k)\rceil}^{G_0} \left(2^{g_0}\right)^{\alpha_0} [\![I^{g_0}_{a_0,b_0}(X)]\!] + \mathcal{C}_0^\dagger \cdot X^{-\eta\theta\frac{s-2k}{s-k}}$$

and

$$\begin{aligned} \mathcal{C}_0 &= 2^{\frac{3}{2}k^2+\frac{11}{2}k-1} k^{\frac{1}{2}k^2+\frac{19}{6}k-2-D} \left(1+\frac{1}{X^\theta}\right)^{2(s-m_0)}, \\ \mathcal{C}_0^\dagger &= 2^{k^2+\frac{11}{3}k} k^{\frac{1}{2}k^2+\frac{19}{6}k-2-D}, \\ \alpha_0 &= 2(k-1)-2(m_0-k). \end{aligned}$$

It is evident that we gave up some saving in the error term E_0. This is because this is the maximal amount of power saving we are able get in the error term E_1 of the next iteration.

We further define

$$\begin{aligned} \Psi_n =& \mathcal{C}_n \left(X^{-\eta\theta}\right)^{\frac{s-2k}{s-k}\sum_{i=0}^{n-1} b_i \left(\frac{k}{s-k}\right)^i} \left(\sum_{g_n=\lceil\log_2(2k)\rceil}^{G_n} \left(2^{g_n}\right)^{\alpha_n} [\![I^{g_n}_{a_n,b_n}(X)]\!]^{\frac{k}{s-k}}\right)^{\left(\frac{k}{s-k}\right)^{n-1}} \\ &+ \mathcal{C}_n^\dagger \cdot X^{-\eta\theta\frac{s-2k}{s-k}}, \end{aligned} \tag{5.46}$$

for $n = 1, \ldots, D$, where

$$\alpha_n = \begin{cases} 2(k-1)-2(m_0-k), & n = 0, \\ 2(k-1)-(2s-\frac{1}{2}k(k+1)+\eta)\left(\frac{m_n}{s}-\frac{k^2}{s(s-k)}\right), & n = 1,\ldots,D, \end{cases}$$

and $\mathcal{C}_n, \mathcal{C}_n^\dagger$ are some constants, which are going to be defined recursively in (5.58) and (5.59). We now use Propositions 5.3.13 and 5.3.15 to prove the following proposition.

Proposition 5.3.17. *With the notation as above and the assumptions mentioned at the beginning of this section, we have*

$$\Psi_n \leq \log_2(2X)^{\left(\frac{k}{s-k}\right)^n} \cdot \Psi_{n+1}, \quad \forall n = 0, \ldots, D-1. \tag{5.47}$$

Proof. As the cases $n = 0$ and $n \geq 1$ are quite similar, we will consider them at the same time. Because of the Inequality (5.45) and because $n \leq D-1$ implies $1 \geq k^2 b_n \theta > 0$, we are able to apply Proposition 5.3.13 to $[\![I^{g_n}_{a_n,b_n}(X)]\!]$ and get

$$[\![I^{g_n}_{a_n,b_n}(X)]\!] \leq C_{n+1} \cdot G_{n+1} \sum_{g_{n+1}=\lceil \log_2(2k) \rceil}^{G_{n+1}} \left(2^{g_{n+1}}\right)^{2(k-1)} [\![K^{g_n, g_{n+1}}_{a_n,b_n;m_{n+1}}(X)]\!] + G_{n+1} \cdot E_{n+1},$$

where

$$C_{n+1} = 2^{4m_{n+1}-2k+2} \cdot k^{2m_{n+1}} \tag{5.48}$$

and

$$\begin{aligned} E_{n+1} =& 2^{2m_{n+1}+1} \cdot k^{2m_{n+1}+1} \cdot \left(2^{g_n}\right)^{-\frac{k}{s}\left(2s - \frac{1}{2}k(k+1)+\eta\right)} \\ & \cdot \left(2^{G_{n+1}}\right)^{2(k-1) - \frac{m_{n+1}}{s}\left(2s-\frac{1}{2}k(k+1)+\eta\right)} \left(X^{\theta}\right)^{\frac{1}{2}k(k+1)\frac{s-k}{s}(b_n - a_n)} \left(X^{-\eta\theta}\right)^{\frac{k}{s}a_n + \frac{s-k}{s}b_n}. \end{aligned} \tag{5.49}$$

In the first sum, we further make use of Proposition 5.3.15, which gives

$$\begin{aligned} [\![K^{g_n, g_{n+1}}_{a_n,b_n;m_{n+1}}(X)]\!] \leq & C'_{n+1} \cdot \left(2^{g_n}\right)^{-k+\frac{1}{2}k(k-1)} \left(2^{g_{n+1}}\right)^{-\left(2s-\frac{1}{2}k(k+1)+\eta\right)\left(\frac{m_{n+1}}{s} - \frac{k^2}{s(s-k)}\right)} \\ & \cdot [\![I^{g_{n+1}}_{a_{n+1},b_{n+1}}(X)]\!]^{\frac{k}{s-k}} \cdot \left(X^{-\eta\theta}\right)^{\frac{s-2k}{s-k}b_n}, \end{aligned}$$

where

$$C'_{n+1} = 2^{\frac{1}{2}k(k+1)+5} \cdot e^{\frac{1}{4}k(3k-2)} \cdot k^{-\frac{1}{2}k(k-2)+1} \cdot (s-k+k^2)^k. \tag{5.50}$$

In the case of $n = 0$, we arrive at the inequality

$$\begin{aligned} \Psi_0 \leq & C_1 C'_1 \mathcal{C}_0 G_1 \left(X^{-\eta\theta}\right)^{\frac{s-2k}{s-k}b_0} \sum_{g_0=\lceil \log_2(2k) \rceil}^{G_0} \left(2^{g_0}\right)^{\alpha_0 - k + \frac{1}{2}k(k-1)} \sum_{g_1=\lceil \log_2(2k) \rceil}^{G_1} \left(2^{g_1}\right)^{\alpha_1} [\![I^{g_1}_{a_1,b_1}(X)]\!]^{\frac{k}{s-k}} \\ & + \mathcal{C}_0 \sum_{g_0=\lceil \log_2(2k) \rceil}^{G_0} \left(2^{g_0}\right)^{\alpha_0} G_1 E_1 + \mathcal{C}_0^\dagger \cdot X^{-\eta\theta\frac{s-2k}{s-k}}. \end{aligned} \tag{5.51}$$

For $n \geq 1$, we further use the elementary inequality $(x+y)^r \leq x^r + y^r$ twice, which holds for $x, y \geq 0$ and $0 \leq r \leq 1$, and arrive at the inequality

$$\begin{aligned}
\Psi_n \leq & \mathcal{C}_n \left(X^{-\eta\theta}\right)^{\frac{s-2k}{s-k}\sum_{i=0}^{n-1} b_i \left(\frac{k}{s-k}\right)^i} \left[\left(X^{-\eta\theta}\right)^{\frac{s-2k}{s-k} b_n \left(\frac{k}{s-k}\right)} \sum_{g_n=\lceil \log_2(2k)\rceil}^{G_n} \left(2^{g_n}\right)^{\alpha_n + \frac{k}{s-k}\left(-k+\frac{1}{2}k(k-1)\right)} \right. \\
& \cdot \left(C_{n+1} C'_{n+1} G_{n+1} \sum_{g_{n+1}=\lceil \log_2(2k)\rceil}^{G_{n+1}} \left(2^{g_{n+1}}\right)^{\alpha_{n+1}} [\![I^{g_{n+1}}_{a_{n+1},b_{n+1}}(X)]\!]^{\frac{k}{s-k}} \right)^{\frac{k}{s-k}} \\
& \left. + \sum_{g_n=\lceil \log_2(2k)\rceil}^{G_n} \left(2^{g_n}\right)^{\alpha_n} \left(G_{n+1} E_{n+1}\right)^{\frac{k}{s-k}} \right]^{\left(\frac{k}{s-k}\right)^{n-1}} + \mathcal{C}^{\dagger}_n X^{-\eta\theta\frac{s-2k}{s-k}} \\
\leq & \mathcal{C}_n \left(X^{-\eta\theta}\right)^{\frac{s-2k}{s-k}\sum_{i=0}^{n} b_i \left(\frac{k}{s-k}\right)^i} \left[\sum_{g_n=\lceil \log_2(2k)\rceil}^{G_n} \left(2^{g_n}\right)^{\alpha_n + \frac{k}{s-k}\left(-k+\frac{1}{2}k(k-1)\right)} \right. \\
& \left. \cdot \left(C_{n+1} C'_{n+1} G_{n+1} \sum_{g_{n+1}=\lceil \log_2(2k)\rceil}^{G_{n+1}} \left(2^{g_{n+1}}\right)^{\alpha_{n+1}} [\![I^{g_{n+1}}_{a_{n+1},b_{n+1}}(X)]\!]^{\frac{k}{s-k}} \right)^{\frac{k}{s-k}} \right]^{\left(\frac{k}{s-k}\right)^{n-1}} \\
& + \mathcal{C}_n \left(X^{-\eta\theta}\right)^{\frac{s-2k}{s-k}} \left[\sum_{g_n=\lceil \log_2(2k)\rceil}^{G_n} \left(2^{g_n}\right)^{\alpha_n} \left(G_{n+1} E_{n+1}\right)^{\frac{k}{s-k}} \right]^{\left(\frac{k}{s-k}\right)^{n-1}} + \mathcal{C}^{\dagger}_n X^{-\eta\theta\frac{s-2k}{s-k}}.
\end{aligned} \tag{5.52}$$

Next, we show that the exponent of 2^{g_n} is at most -1 if $n = 0$ and $-\frac{1}{3}$ otherwise. First, we consider the case $n = 0$. There, we have

$$2(k-1) - 2(m_0 - k) - k + \frac{1}{2}k(k-1) \leq -1 \Leftrightarrow \frac{1}{4}k(k+1) + k - \frac{1}{2} \leq m_0,$$

which is true. Now, we analyse the case when $n > 0$. There, we have to bound

$$2(k-1) - \left(2s - \frac{1}{2}k(k+1) + \eta\right)\left(\frac{m_n}{s} - \frac{k^2}{s(s-k)}\right) + \frac{k}{s-k}\left[-k + \frac{1}{2}k(k-1)\right].$$

Since $m_n \geq 1 \geq \frac{k^2}{s-k}$, we only make the expression bigger when replacing $2s - \frac{1}{2}k(k+1) + \eta$ by $\frac{3}{2}s$ as the latter is smaller. Thus, we are left to bound

$$\begin{aligned}
& 2(k-1) - \frac{3}{2}\left(m_n - \frac{k^2}{s-k}\right) + \frac{k}{s-k}\left[-k + \frac{1}{2}k(k-1)\right] \\
= & 2(k-1) - \frac{3}{2}m_n + \frac{k}{s-k}\left[\frac{3}{2}k - k + \frac{1}{2}k(k-1)\right].
\end{aligned}$$

Now, we have $\frac{3}{2}k - k + \frac{1}{2}k(k-1) \geq 0$ and hence the expression gets bigger when we replace s by $k^2 + k$ as the latter is smaller. We are left to deal with

$$\frac{5}{2}k - 2 - \frac{3}{2}m_n.$$

It suffices to have
$$\frac{5}{2}k-2-\frac{3}{2}m_n \le -\frac{1}{3} \Leftrightarrow \frac{5}{3}k-\frac{10}{9} \le m_n,$$
which is true. Therefore, we conclude
$$\begin{aligned} \sum_{g_0=\lceil \log_2(2k)\rceil}^{G_0} \left(2^{g_0}\right)^{\alpha_0-k+\frac{1}{2}k(k-1)} &\le \frac{1}{k}, \\ \sum_{g_n=\lceil \log_2(2k)\rceil}^{G_n} \left(2^{g_n}\right)^{\alpha_n+\frac{k}{s-k}\left(-k+\frac{1}{2}k(k-1)\right)} &\le \frac{4}{k^{\frac{1}{3}}} \quad \forall n\ge 1. \end{aligned} \tag{5.53}$$

Now, we turn our attention to the analysis of the error term; i.e. the terms involving E_{n+1}. Let us consider the exponent of 2^{g_n} first. For $n=0$, the exponent is
$$\begin{aligned} 2(k-1)-2(m_0-k)-\frac{k}{s}\left(2s-\frac{1}{2}k(k+1)+\eta\right) &\le 2(k-1)-2(m_0-k)-\frac{3}{2}k \\ &\le -\frac{1}{2}k^2-\frac{2}{3}k+\frac{2}{3} \le -1. \end{aligned}$$

Thus, we have
$$\sum_{g_0=\lceil \log_2(2k)\rceil}^{G_0} \left(2^{g_0}\right)^{\alpha_0-\frac{k}{s}\left(2s-\frac{1}{2}k(k+1)+\eta\right)} \le 2(2k)^{-\frac{1}{2}k^2-\frac{2}{3}k+\frac{2}{3}}. \tag{5.54}$$

For $n\ge 1$, the exponent is
$$\begin{aligned} 2(k-1)-\left(2s-\frac{1}{2}k(k+1)+\eta\right)&\left(\frac{m_n}{s}-\frac{k^2}{s(s-k)}\right)-\frac{k}{s-k}\frac{k}{s}\left(2s-\frac{1}{2}k(k+1)+\eta\right) \\ &= 2(k-1)-\frac{m_n}{s}\left(2s-\frac{1}{2}k(k+1)+\eta\right) \\ &\le 2(k-1)-\frac{3}{2}m_n \\ &\le -\frac{1}{2}k-1. \end{aligned}$$

Thus, we get for $n\ge 1$, that
$$\sum_{g_n=\lceil \log_2(2k)\rceil}^{G_n} \left(2^{g_n}\right)^{\alpha_n-\frac{k}{s-k}\frac{k}{s}\left(2s-\frac{1}{2}k(k+1)+\eta\right)} \le 2(2k)^{-\frac{1}{2}k-1}. \tag{5.55}$$

Now, we consider the power of X in the error term E_{n+1}; i.e. we are having a detailed look at
$$\left(2^{G_{n+1}}\right)^{2(k-1)-\frac{m_{n+1}}{s}\left(2s-\frac{1}{2}k(k+1)+\eta\right)}\left(X^{\theta}\right)^{\frac{1}{2}k(k+1)\frac{s-k}{s}(b_n-a_n)}\left(X^{-\eta\theta}\right)^{\frac{k}{s}a_n+\frac{s-k}{s}b_n}.$$

For $n=0$, we bound
$$\left(X^{-\eta\theta}\right)^{\frac{k}{s}a_0+\frac{s-k}{s}b_0} \le X^{-\eta\theta\frac{s-2k}{s-k}}$$

and for $n \geq 1$ we bound trivially

$$\left(X^{-\eta\theta}\right)^{\frac{k}{s}a_n+\frac{s-k}{s}b_n} \leq 1.$$

For the rest, we use the inequality $G_{n+1} \geq k^{n+1}\theta\log_2(X)-1$ and find

$$\begin{aligned}
&\left(2^{G_{n+1}}\right)^{2(k-1)-\frac{m_{n+1}}{s}(2s-\frac{1}{2}k(k+1)+\eta)}\left(X^{\theta}\right)^{\frac{1}{2}k(k+1)\frac{s-k}{s}(b_n-a_n)} \\
&\qquad \leq \left(\frac{X^{\theta k^{n+1}}}{2}\right)^{2(k-1)-\frac{3}{2}m_{n+1}}\left(X^{\theta k^{n+1}}\right)^{\frac{1}{2}(k+1)} \\
&\qquad \leq 2^{\frac{1}{2}k+1}\left(X^{\theta k^{n+1}}\right)^{-\frac{1}{2}} \\
&\qquad \leq 2^{\frac{1}{2}k+1}2^{-\frac{1}{2}k^{n+1}} \\
&\qquad \leq \begin{cases} 2, & n=0, \\ 1, & n\geq 1. \end{cases}
\end{aligned}$$

The latter seems inefficient, but one has to consider that the $((s-k)/k)^n$-th root will be taken of it in due course. Hence, we have

$$\begin{aligned}
&\left(2^{G_{n+1}}\right)^{2(k-1)-\frac{m_{n+1}}{s}(2s-\frac{1}{2}k(k+1)+\eta)}\left(X^{\theta}\right)^{\frac{1}{2}k(k+1)\frac{s-k}{s}(b_n-a_n)}\left(X^{-\eta\theta}\right)^{\frac{k}{s}a_n+\frac{s-k}{s}b_n} \\
&\qquad\qquad \leq \begin{cases} 2X^{-\eta\theta\frac{s-2k}{s-k}}, & n=0, \\ 1, & n\geq 1. \end{cases}
\end{aligned} \tag{5.56}$$

Lastly, we have

$$G_{n+1} \leq k^{n-D}\log_2(2X) \leq \begin{cases} k^{-D}\log_2(2X), & n=0, \\ k^{-1}\log_2(2X), & n\geq 1. \end{cases} \tag{5.57}$$

By collecting all of the previous analysis, we have proven (5.47). We go through this one step at a time. For $n=0$, we combine (5.51) with (5.53) and (5.57); this gives us the main term and $\mathcal{C}_1$ as in (5.58). For the error term, we combine (5.51) with (5.49), (5.54), (5.56) and (5.57). This yields the following admissible value for $\mathcal{C}_1^{\dagger}$:

$$\begin{aligned}
\mathcal{C}_1 &= \mathcal{C}_0 \cdot C_1C_1' \cdot k^{-D}\cdot k^{-1}, \\
\mathcal{C}_1^{\dagger} &= \mathcal{C}_0^{\dagger} + \mathcal{C}_0 \cdot 2^{2m_1+1} \cdot k^{2m_1+1} \cdot 2(2k)^{-\frac{1}{2}k^2-\frac{2}{3}k+\frac{2}{3}} \cdot k^{-D} \cdot 2 \\
&\leq \mathcal{C}_0^{\dagger} + \mathcal{C}_0 \cdot 2^{\frac{10}{3}k+1} \cdot k^{\frac{10}{3}k+1} \cdot 2(2k)^{-\frac{1}{2}k^2-\frac{2}{3}k+\frac{2}{3}} \cdot k^{-D} \cdot 2 \\
&\leq \mathcal{C}_0^{\dagger} + \mathcal{C}_0 \cdot 2^{-\frac{1}{2}k^2+\frac{8}{3}k+\frac{11}{3}}k^{-\frac{1}{2}k^2+\frac{8}{3}k+\frac{5}{3}-D}.
\end{aligned} \tag{5.58}$$

For $n = 1, \ldots, D-1$, we combine (5.52) with (5.53) and (5.57); this gives us the main term with $\mathcal{C}_{n+1}$ as in (5.59). For the error term, we combine (5.52) with (5.49), (5.55), (5.56) and (5.57). This yields the following admissible value for $\mathcal{C}_{n+1}^\dagger$:

$$\begin{aligned} \mathcal{C}_{n+1} &= \mathcal{C}_n \cdot \left(\frac{4}{k^{\frac{1}{3}}}\right)^{\left(\frac{k}{s-k}\right)^{n-1}} \left(C_{n+1}C'_{n+1} \cdot k^{-1}\right)^{\left(\frac{k}{s-k}\right)^n}, \\ \mathcal{C}_{n+1}^\dagger &= \mathcal{C}_n^\dagger + \mathcal{C}_n \left(2(2k)^{-\frac{1}{2}k-1}\right)^{\left(\frac{k}{s-k}\right)^{n-1}} \left(2^{2m_{n+1}+1}k^{2m_{n+1}+1}k^{-1}\right)^{\left(\frac{k}{s-k}\right)^n}. \end{aligned} \tag{5.59}$$

We bound $\mathcal{C}_{n+1}^\dagger$ further by

$$\begin{aligned} \mathcal{C}_{n+1}^\dagger &\leq \mathcal{C}_n^\dagger + \mathcal{C}_n \left(\left(2(2k)^{-\frac{1}{2}k-1}\right)^k \cdot 2^{2m_{n+1}+1}k^{2m_{n+1}+1}k^{-1}\right)^{\left(\frac{k}{s-k}\right)^n} \\ &\leq \mathcal{C}_n^\dagger + \mathcal{C}_n \left(2^{-\frac{1}{2}k^2+\frac{10}{3}k+1}k^{-\frac{1}{2}k^2+\frac{7}{3}k}\right)^{\left(\frac{k}{s-k}\right)^n}. \end{aligned} \tag{5.60}$$

□

It remains to estimate Ψ_D. This is done with the help of Proposition 5.3.16 and yields the following proposition.

Proposition 5.3.18. *With the assumptions as in Proposition 5.3.17, we have*

$$\Psi_D \leq \mathcal{C}_{D+1} X^{\frac{k^2(k^2-1)}{2s} b_D \left(\frac{k}{s-k}\right)^D \theta - \eta \frac{s-2k}{s-k} \sum_{i=0}^D b_i \left(\frac{k}{s-k}\right)^i \theta} + \mathcal{C}_D^\dagger \cdot X^{-\eta\theta\frac{s-2k}{s-k}},$$

where

$$\mathcal{C}_{D+1} = \mathcal{C}_D \cdot \left(\frac{4}{k^{\frac{1}{3}}}\right)^{\left(\frac{k}{s-k}\right)^{D-1}} C'_{D+1}{}^{\left(\frac{k}{s-k}\right)^D}$$

and

$$\begin{aligned} C'_{D+1} &= 2^{\frac{1}{2}k(k+5)+4} \cdot e^{\frac{1}{4}k(3k-2)} \cdot k^{-\frac{1}{2}k(k-2)+1} \cdot (s-k+k^2)^k \\ &= C'_D \cdot 2^{2k}. \end{aligned}$$

Proof. The proof follows from Proposition 5.3.16 combined with (5.53) applied to (5.46). □

We are left with estimating the constants. For $n = 1, \ldots, D$, we have from (5.48)

$$C_n = 2^{4m_n-2k+2} \cdot k^{2m_n} \leq 2^{\frac{14}{3}k+2} \cdot k^{\frac{10}{3}k}$$

and from (5.50)

$$C'_n = 2^{\frac{1}{2}k(k+1)+5} \cdot e^{\frac{1}{4}k(3k-2)} \cdot k^{-\frac{1}{2}k(k-2)+1} \cdot (s-k+k^2)^k.$$

By inserting this into Proposition 5.3.17 and using the Definition (5.59), we get[a]

$$\begin{aligned}\mathcal{C}_n \leq & 2^{\frac{3}{2}k^2+\frac{11}{2}k-1} k^{\frac{1}{2}k^2+\frac{19}{6}k-2-2D}\left(1+\frac{1}{X^\theta}\right)^{2(s-m_0)} \cdot \left(\frac{4}{k^{\frac{1}{3}}}\right)^{\sum_{i=0}^{n-2}\left(\frac{k}{s-k}\right)^i} \\ & \cdot \left(2^{\frac{1}{2}k^2+\frac{31}{6}k+7} e^{\frac{3}{4}k^2-\frac{1}{2}k} k^{-\frac{1}{2}k^2+\frac{19}{3}k}(\lambda+1)^k\right)^{\sum_{i=0}^{n-1}\left(\frac{k}{s-k}\right)^i}\end{aligned} \tag{5.61}$$

for $n = 1, \ldots, D$. We further have

$$2^{2k} \leq 2^{\frac{14}{3}k+2} \cdot k^{\frac{10}{2}k} \cdot k^{-1}.$$

By using these two inequalities with Proposition 5.3.18, we get

$$\begin{aligned}\mathcal{C}_{D+1} \leq & 2^{\frac{3}{2}k^2+\frac{11}{2}k-1} k^{\frac{1}{2}k^2+\frac{19}{6}k-2-2D}\left(1+\frac{1}{X^\theta}\right)^{2(s-m_0)} \cdot \left(\frac{4}{k^{\frac{1}{3}}}\right)^{\sum_{i=0}^{D-1}\left(\frac{k}{s-k}\right)^i} \\ & \cdot \left(2^{\frac{1}{2}k^2+\frac{31}{6}k+7} e^{\frac{3}{4}k^2-\frac{1}{2}k} k^{-\frac{1}{2}k^2+\frac{19}{3}k}(\lambda+1)^k\right)^{\sum_{i=0}^{D}\left(\frac{k}{s-k}\right)^i}.\end{aligned} \tag{5.62}$$

We now turn our attention to bounding $\mathcal{C}_n^\dagger$. We continue the estimation (5.58) for $\mathcal{C}_1^\dagger$:

$$\begin{aligned}\mathcal{C}_1^\dagger \leq & 2^{k^2+\frac{11}{3}k} k^{\frac{1}{2}k^2+\frac{19}{6}k-2-D} + 2^{\frac{3}{2}k^2+\frac{11}{2}k-1} k^{\frac{1}{2}k^2+\frac{19}{6}k-2-D}\left(1+\frac{1}{X^\theta}\right)^{2(s-m_0)} \\ & \cdot 2^{-\frac{1}{2}k^2+\frac{8}{3}k+\frac{11}{3}} k^{-\frac{1}{2}k^2+\frac{8}{3}k+\frac{5}{3}-D} \\ \leq & 2^{\frac{3}{2}k^2+\frac{11}{2}k-1} k^{\frac{1}{2}k^2+\frac{19}{6}k-2-D}\left(1+\frac{1}{X^\theta}\right)^{2(s-m_0)} \\ & \cdot \left(2^{-\frac{1}{2}k^2-\frac{11}{6}k+1} + 2^{-\frac{1}{2}k^2+\frac{8}{3}k+\frac{11}{3}} k^{-\frac{1}{2}k^2+\frac{8}{3}k+\frac{5}{3}-D}\right).\end{aligned} \tag{5.63}$$

By using induction on (5.60) with (5.61) and (5.63) as a base, we further find

$$\begin{aligned}\mathcal{C}_n^\dagger \leq & 2^{\frac{3}{2}k^2+\frac{11}{2}k-1} k^{\frac{1}{2}k^2+\frac{19}{6}k-2-D}\left(1+\frac{1}{X^\theta}\right)^{2(s-m_0)} \\ & \cdot \Bigg[2^{-\frac{1}{2}k^2-\frac{11}{6}k+1} + 2^{-\frac{1}{2}k^2+\frac{8}{3}k+\frac{11}{3}} k^{-\frac{1}{2}k^2+\frac{8}{3}k+\frac{2}{3}} \\ & + k^{-1}\sum_{i=2}^{n}\left(\left(\frac{4}{k^{\frac{1}{3}}}\right)^{\sum_{j=0}^{i-3}\left(\frac{k}{s-k}\right)^j}\left(2^{\frac{1}{2}k^2+\frac{31}{6}k+7} e^{\frac{3}{4}k^2-\frac{1}{2}k} k^{-\frac{1}{2}k^2+\frac{19}{3}k}(\lambda+1)^k\right)^{\sum_{j=0}^{i-1}\left(\frac{k}{s-k}\right)^j}\right)\Bigg]\end{aligned} \tag{5.64}$$

for $n = 1, \ldots, D$, where we have made use of the inequality

$$2^{-\frac{1}{2}k^2+\frac{10}{3}k+1} k^{-\frac{1}{2}k^2+\frac{7}{3}k} \leq 2^{\frac{1}{2}k^2+\frac{31}{6}k+7} e^{\frac{3}{4}k^2-\frac{1}{2}k} k^{-\frac{1}{2}k^2+\frac{19}{3}k}(\lambda+1)^k$$

and $k^{-D} \leq k^{-1}$. Let us now tame the inequality (5.64). We have for any $n \in \mathbb{Z}$

$$\left(\frac{4}{k^{\frac{1}{3}}}\right)^{\sum_{i=0}^{n}\left(\frac{k}{s-k}\right)^i} \leq \max\left\{1, \sup_{k\geq 3}\left(\frac{4}{k^{\frac{1}{3}}}\right)^{\frac{k}{k-1}}\right\} \leq \frac{2^3}{\sqrt{3}} \leq 2k$$

---

a Here and throughout this section, $\sum_{i=0}^{-l}$ denotes the empty sum for any $l > 0$ and equals 0.

as the latter is a decreasing function in k. Let $\mathcal{M}$ denote the maximum of the quantities

$$\begin{aligned} & 2^{-\frac{1}{2}k^2-\frac{11}{6}k}, \\ & 2^{-\frac{1}{2}k^2+\frac{8}{3}k+\frac{8}{3}}k^{-\frac{1}{2}k^2+\frac{8}{3}k+\frac{2}{3}}, \\ & \left(2^{\frac{1}{2}k^2+\frac{31}{6}k+7}e^{\frac{3}{4}k^2-\frac{1}{2}k}k^{-\frac{1}{2}k^2+\frac{19}{3}k}(\lambda+1)^k\right)^\gamma, \quad \gamma \in \left\{1, \frac{s-k}{s-2k}\right\}, \end{aligned} \tag{5.65}$$

then we have

$$\begin{aligned} \mathcal{C}_D^\dagger &\leq 2^{\frac{3}{2}k^2+\frac{11}{2}k}k^{\frac{1}{2}k^2+\frac{19}{6}k-2-D}\left(1+\frac{1}{X^\theta}\right)^{2(s-m_0)} \cdot (D+1)\mathcal{M} \\ &\leq 2^{\frac{3}{2}k^2+\frac{11}{2}k}k^{\frac{1}{2}k^2+\frac{19}{6}k-2}\left(1+\frac{1}{X^\theta}\right)^{2(s-m_0)} \mathcal{M}. \end{aligned}$$

Returning to (5.62), we also have

$$\mathcal{C}_{D+1} \leq 2^{\frac{3}{2}k^2+\frac{11}{2}k}k^{\frac{1}{2}k^2+\frac{19}{6}k-2}\left(1+\frac{1}{X^\theta}\right)^{2(s-m_0)} \mathcal{M}.$$

We immediately see that the middle expression in (5.65) is dominated by the latter one. We also make use of the inequality $\lambda+1 \leq k^2$ and hence $\mathcal{M}$ is at most $\mathcal{M}_0$, where we recall (5.11):

$$\mathcal{M}_0 = \max_{\gamma \in \{1, \frac{s-k}{s-2k}\}} \left\{ \left(2^{\frac{1}{2}k^2+\frac{31}{6}k+7}e^{\frac{3}{4}k^2-\frac{1}{2}k}k^{-\frac{1}{2}k^2+\frac{25}{3}k}\right)^\gamma, 2^{-\frac{1}{2}k^2-\frac{11}{6}k} \right\}.$$

We conclude the following proposition.

Proposition 5.3.19. *Let $s, k \in \mathbb{N}$ with $k \geq 3$ and $2\log(k) \geq \lambda = \frac{s-k}{k^2} \geq 1$. Further, let $D \geq 1$ be an integer and set $\theta = k^{-(D+1)}$. Assume that*

$$J_{s,k}(X) \leq C\log_2(2X)^\delta X^{2s-\frac{1}{2}k(k+1)+\eta} \quad \forall X \geq 1,$$

for some $0 \leq \delta$ and $0 < \eta \leq \frac{1}{2}k(k+1)$. Then, we have

$$J_{s,k}(X) \leq C'\log_2(2X)^{\delta+\frac{2\lambda k-1}{\lambda k-1}} X^{2s-\frac{1}{2}k(k+1)+\eta}\left(X^{\Delta\theta} + X^{-\eta\theta\frac{s-2k}{s-k}}\right) \quad \forall X^\theta \geq 2,$$

where

$$\begin{aligned} \Delta &= \frac{k^2(k^2-1)}{2s}\lambda^{-D} - \eta\frac{s-2k}{s-k}\sum_{i=0}^{D}\lambda^{-i}, \\ C' &= C \cdot 2^{\frac{3}{2}k^2+\frac{11}{2}k}k^{\frac{1}{2}k^2+\frac{19}{6}k-2}\left(1+\frac{1}{X^\theta}\right)^{2(s-m_0)}\mathcal{M}_0, \end{aligned}$$

with m_0 as in (5.44) and where $\mathcal{M}_0$ is defined as in (5.11).

Now, we want to bring Proposition 5.3.19 into a shape which one can iterate easily. For this matter, we want to optimise our gain in the exponent. The optimal choice of D is in general not an easy problem and leads to complications in further calculations. Nevertheless, there is a reasonable exponent gain one can achieve, namely $-\eta\theta\frac{s-2k}{s-k}$. This is reasonable because if λ is close to 1 all terms are of almost equal size and if λ is large the positive term gets very small and thus can be handled by the tail sum.

Let us first assume $\lambda > 1$. Then,

$$\Delta = -\eta\frac{s-2k}{s-k} + \frac{k^2(k^2-1)}{2s}\lambda^{-D} - \eta\frac{s-2k}{s-k}\frac{\lambda^{-1}(1-\lambda^{-D})}{1-\lambda^{-1}},$$

and we would like $\Delta \le -\eta\frac{s-2k}{s-k}$. Thus, we require

$$\left(\frac{k^2(k^2-1)}{2s} + \eta\frac{s-2k}{s-k}\frac{\lambda^{-1}}{1-\lambda^{-1}}\right)\lambda^{-D} \le \eta\frac{s-2k}{s-k}\frac{\lambda^{-1}}{1-\lambda^{-1}}$$

or

$$\frac{k^2}{2\eta}\frac{k^2-1}{s}\frac{s-k}{s-2k}\frac{\lambda-1}{\lambda} + 1 \le \lambda^D.$$

Now, we have

$$\frac{k^2-1}{s}\frac{s-k}{s-2k}\frac{\lambda-1}{\lambda} = \frac{k^2-1}{\lambda k+1}\frac{\lambda}{\lambda k-1}\frac{\lambda-1}{\lambda} = \frac{(\lambda-1)(k^2-1)}{\lambda^2k^2-1} \le \frac{\lambda-1}{\lambda^2}.$$

Hence, it suffices to have

$$D \ge \frac{\log\left(\frac{k^2}{2\eta}\frac{\lambda-1}{\lambda^2}+1\right)}{\log(\lambda)}. \tag{5.66}$$

In the case $\lambda = 1$, i.e. $s = k(k+1)$, one needs

$$\frac{k^2(k^2-1)}{2s} - \eta\frac{s-2k}{s-k}D \le 0 \Leftrightarrow D \ge \frac{k^2}{2\eta},$$

which is recovered from (5.66) in the limit as $\lambda \to 1^+$.

We are now able to balance the two inequalities in Proposition 5.3.19. We make the choice $X_0^\theta = 4k$ and use the trivial inequality for $1 \le X \le X_0$ and the new inequality for $X \ge X_0$. Thus, we have for $1 \le X \le X_0$:

$$\begin{aligned} J_{s,k}(X) &\le C\log_2(2X)^\delta X^{2s-\frac{1}{2}k(k+1)+\eta} \\ &\le C\log_2(2X)^\delta X^{2s-\frac{1}{2}k(k+1)+\eta}\left(X_0^{\eta\theta}\cdot X^{-\eta\theta\frac{s-2k}{s-k}}\right) \\ &\le C\cdot 2^{k^2+k}k^{\frac{1}{2}k^2+\frac{1}{2}k}\cdot\log_2(2X)^{\delta+\frac{2\lambda k-1}{\lambda k-1}}X^{2s-\frac{1}{2}k(k+1)+\eta}\cdot X^{-\eta\theta\frac{s-2k}{s-k}}, \end{aligned} \tag{5.67}$$

where we have made use of $\eta \le \frac{1}{2}k(k+1)$. For $X \ge X_0$, we further need to estimate

$$\left(1+\frac{1}{X_0^\theta}\right)^{2(s-m_0)} \le \left(1+\frac{1}{4k}\right)^{2\lambda k^2} \le e^{\frac{1}{2}\lambda k} \le k^k.$$

Thus, in this case, we get

$$\begin{aligned} J_{s,k}(X) \le & C \cdot 2^{\frac{3}{2}k^2+\frac{11}{2}k+1} k^{\frac{1}{2}k^2+\frac{25}{6}k-2} \cdot \mathcal{M}_0 \\ & \cdot \log_2(2X)^{\delta+\frac{2\lambda k-1}{\lambda k-1}} X^{2s-\frac{1}{2}k(k+1)+\eta} \cdot X^{-\eta\theta\frac{s-2k}{s-k}}. \end{aligned} \tag{5.68}$$

By comparing the two constants in (5.67) and (5.68), we find that the latter is larger and thus we conclude the proof of Theorem 5.3.1.

5.3.6 *Final Upper Bounds*

In this section, we consider a more general system of equations

$$\sum_{i=1}^{l} x_i^j - \sum_{i=l+1}^{s} x_i^j = N_j, \quad (j = 1, \dots, k),$$

with integers $1 \le x_i \le X$. Let $I_{s,k,l}(\boldsymbol{N}; X)$ denote its counting function. We shall use a Hardy–Littlewood dissection into major and minor arcs to establish an asymptotic formula

$$I_{s,k,l}(\boldsymbol{N}; X) \sim \mathfrak{S}_{s,k,l}(\boldsymbol{N}) \mathcal{J}_{s,k,l}(\boldsymbol{N}) X^{s-\frac{1}{2}k(k+1)}$$

with an effective error term, where $\mathfrak{S}_{s,k,l}(\boldsymbol{N})$ and $\mathcal{J}_{s,k,l}(\boldsymbol{N})$ are the singular series and the singular integral, which are given by

$$\mathfrak{S}_{s,k,l}(\boldsymbol{N}) = \sum_{q_1,\dots,q_n=1}^{\infty} \sum_{\substack{\boldsymbol{a} \bmod \boldsymbol{q} \\ (a_i,q_i)=1, i=1,\dots,k}} (q_1 \cdot \dots \cdot q_k)^{-s} S_{\boldsymbol{q}}(\boldsymbol{a})^l \overline{S_{\boldsymbol{q}}(\boldsymbol{a})}^{s-l} e\left(-\sum_{j=1}^{k} \frac{a_j N_j}{q_j}\right)$$

and

$$\mathcal{J}_{s,k,l}(\boldsymbol{N}) = \int_{\mathbb{R}^k} I(\boldsymbol{\beta})^l \overline{I(\boldsymbol{\beta})}^{s-l} e\left(-\sum_{j=1}^{k} \frac{\beta_j N_j}{X^j}\right) d\boldsymbol{\beta},$$

where

$$S_{\boldsymbol{q}}(\boldsymbol{a}) = \sum_{n=1}^{q} e\left(\sum_{j=1}^{k} \frac{a_j n^j}{q_j}\right) \text{ and } I(\boldsymbol{\beta}) = \int_0^1 e\left(\sum_{j=1}^{k} \beta_j x^j\right) dx.$$

We achieve this by using a good enough estimate for $J_{s,k}(X)$ in the minor arcs, which we will get by iterating Theorem 5.3.1. To make our life simpler, we restrict to the case $\lambda > 1$ and think of λ as fixed as in this case we see that D only grows logarithmically in $\frac{k^2}{2\eta}$, which, in return, makes the constant smaller.

We will iterate Theorem 5.3.1 as follows. We fix D and iterate as many times as needed till we get an exponent η that is too small to apply the theorem with the choice of D we fixed. For this purpose, we need to reverse engineer the inequality (5.66). We have

$$\begin{aligned}\frac{k^2}{2\eta}\cdot\frac{\lambda-1}{\lambda^2}+1 &= \frac{k^2}{2\eta}\cdot\frac{\lambda-1}{\lambda^2}\left(1+\frac{2\eta}{k^2}\cdot\frac{\lambda^2}{\lambda-1}\right)\\ &\le \frac{k^2}{2\eta}\cdot\frac{\lambda-1}{\lambda^2}\left(1+\frac{k(k+1)}{k^2}\cdot\frac{\lambda^2}{\lambda-1}\right)\\ &\le \frac{k^2}{2\eta}\cdot\frac{\lambda-1}{\lambda^2}\left(\frac{k(k+1)}{k^2}\cdot\frac{\lambda^2+\lambda-1}{\lambda-1}\right)\\ &= \frac{k(k+1)}{2\eta}\cdot\frac{\lambda^2+\lambda-1}{\lambda^2}\\ &\le \frac{5}{4}\cdot\frac{k(k+1)}{2\eta}\end{aligned} \tag{5.69}$$

as $\lambda \le \frac{1}{4}\lambda^2+1$ by the inequality between the arithmetic and geometric mean. Thus, we are able to apply Theorem 5.3.1 as long as

$$\eta \ge \frac{5}{4}\cdot\frac{k(k+1)}{2\lambda^D},$$

which immediately leads to the following proposition.

Proposition 5.3.20. *Let $s,k,D\in\mathbb{N}$ with $k\ge 3$ and $2\log(k)\ge\lambda=\frac{s-k}{k^2}>1$. Assume that*

$$J_{s,k}(X)\le C\log_2(2X)^{\delta}X^{2s-\frac{1}{2}k(k+1)+\eta}\quad \forall X\ge 1,$$

for some $0\le\delta$ and $0<\eta\le\max\left\{\frac{1}{2}k(k+1),\frac{5}{4}\cdot\frac{k(k+1)}{2\lambda^{D-1}}\right\}$. Then, we have

$$\begin{aligned}J_{s,k}(X)\le &C\left[2^{\frac{3}{2}k^2+\frac{11}{2}k+1}k^{\frac{1}{2}k^2+\frac{25}{6}k-2}\mathcal{M}_0\cdot\log_2(2X)^{\frac{2\lambda k-1}{\lambda k-1}}\right]^{\log(\lambda)\frac{\lambda k}{\lambda k-1}k^{D+1}+1}\\ &\cdot\log_2(2X)^{\delta}X^{2s-\frac{1}{2}k(k+1)+\eta'},\quad\forall X\ge 1,\end{aligned}$$

for some $\eta'<\frac{5}{4}\cdot\frac{k(k+1)}{2\lambda^D}$ and where $\mathcal{M}_0$ as defined in (5.11).

Proof. If $\eta<\frac{5}{4}\cdot\frac{k(k+1)}{2\lambda^D}$, then the statement is trivial. Otherwise, we are able to apply Theorem 5.3.1 and we receive an inequality with

$$\eta'=\eta\left(1-\frac{1}{k^{D+1}}\frac{\lambda k-1}{\lambda k}\right).$$

If $\eta'<\frac{5}{4}\cdot\frac{k(k+1)}{2\lambda^D}$, then we are done otherwise we repeat the process. After at most

$$\left\lceil k^{D+1}\frac{\lambda k}{\lambda k-1}\log(\lambda)\right\rceil\le k^{D+1}\frac{\lambda k}{\lambda k-1}\log(\lambda)+1$$

iterations we are guaranteed to have $\eta'<\frac{5}{4}\cdot\frac{k(k+1)}{2\lambda^D}$ and hence conclude the proof of the proposition. ∎

We get the following corollary immediately.

Corollary 5.3.21. *Let $s, k, D \in \mathbb{N}$ with $k \geq 3$ and $2\log(k) \geq \lambda = \frac{s-k}{k^2} > 1$. Then, we have*

$$J_{s,k}(X) \leq \left[2^{\frac{3}{2}k^2+\frac{11}{2}k+1}k^{\frac{1}{2}k^2+\frac{25}{6}k-2}\mathcal{M}_0 \cdot \log_2(2X)^{\frac{2\lambda k-1}{\lambda k-1}}\right]^{\log(\lambda)\frac{\lambda k}{\lambda k-1}k^2\frac{k^D-1}{k-1}+D} \cdot X^{2s-\frac{1}{2}k(k+1)+\frac{5}{4}\cdot\frac{k(k+1)}{2\lambda^D}}, \quad \forall X \geq 1,$$

where $\mathcal{M}_0$ as defined in (5.11).

The next step is to get an asymptotic formula as well as an upper bound of the right order of magnitude. From now on, we restrict ourselves to the case $\lambda = 2$, i.e. $s = 2k^2 + k$. For this purpose, we follow the argument throughout pages 114 to 132 of [ACK04] and insert Corollary 5.3.21 in the treatment of the minor arcs.

First we bring the estimate in Corollary 5.3.21 into a shape without logarithms. For $X \geq 7$, we have $\log_2(2X) \leq 2\log(X)$. Moreover, we have $\frac{2k}{2k-1} \leq \frac{6}{5}$. Furthermore, we have the inequality

$$\log(X)^\alpha \leq \left(\frac{\alpha}{\beta e}\right)^\alpha X^\beta, \quad \forall \alpha, \beta > 0, X \geq e,$$

as the function $\alpha\log(\log(X)) - \beta\log(X)$ reaches its maximum at $X = e^{\frac{\alpha}{\beta}}$. Hence, we conclude for $X \geq 7$, that

$$\begin{aligned}
&\log_2(2X)^{\frac{4k-1}{2k-1}\left(\log(2)\frac{2k}{2k-1}k^2\frac{k^D-1}{k-1}+D\right)}\\
&\leq \left(2 \cdot \frac{\frac{4k-1}{2k-1}\left(\log(2)\frac{2k}{2k-1}k^2\frac{k^D-1}{k-1}+D\right)}{\frac{5e}{4}\cdot\frac{k(k+1)}{2^{D+1}}}\right)^{\frac{4k-1}{2k-1}\left(\log(2)\frac{2k}{2k-1}k^2\frac{k^D-1}{k-1}+D\right)} X^{\frac{5}{4}\cdot\frac{k(k+1)}{2^{D+1}}}\\
&\leq \left[\frac{2.6\cdot 2^D}{k(k+1)}\left(\frac{6}{5}\log(2)k^2\frac{k^D-1}{k-1}+D\right)\right]^{\frac{11}{5}\left(\frac{6}{5}\log(2)k^2\frac{k^D-1}{k-1}+D\right)} X^{\frac{5}{4}\cdot\frac{k(k+1)}{2^{D+1}}}
\end{aligned}$$

holds. Furthermore, we have

$$\begin{aligned}
\frac{6}{5}\log(2)k^2\frac{k^D-1}{k-1}+D &\leq \frac{6}{5}\log(2)k^2\frac{k^D-1}{k-1}\left(1+\frac{D}{\frac{6}{5}\log(2)k^{D+1}}\right)\\
&\leq \frac{6}{5}\log(2)k^2\frac{k^D-1}{k-1}\left(1+\frac{5}{54\log(2)}\right)\\
&\leq k^2\frac{k^D-1}{k-1}\\
&\leq \frac{3}{2}k^{D+1}
\end{aligned}$$

and

$$\frac{2.6}{k(k+1)}k^2\frac{k^D-1}{k-1} \leq 2.6\cdot\frac{k}{k^2-1}\cdot k^D \leq k^D.$$

Hence, we may conclude that

$$\log_2(2X)^{\frac{4k-1}{2k-1}\left(\log(2)\frac{2k}{2k-1}k^2\frac{k^D-1}{k-1}+D\right)} \leq \left(2^D k^D\right)^{\frac{11}{5}\left(\frac{3}{2}k^{D+1}\right)} X^{\frac{5}{4}\cdot\frac{k(k+1)}{2^{D+1}}}.$$

Hence, by increasing the constant appropriately we are able to have that the dependency on X is only $X^{2s-\frac{1}{2}k(k+1)+\frac{5}{4}\cdot\frac{k(k+1)}{2^D}}$. We now make use of this inequality in the treatment of I_2 in [ACK04] on page 121 with $k_1 = k^2$ and $k_2 = 2k^2+k$. In order to have a power saving, we need

$$\frac{5}{4}\cdot\frac{k(k+1)}{2^D} < k^2\cdot\rho = k^2\cdot\left(8k^2(\log(k)+1.5\log(\log(k))+4.2)\right)^{-1},$$

which is equivalent to

$$10k(k+1)(\log(k)+1.5\log(\log(k))+4.2) < 2^D.$$

Since we have $k+1 \leq \frac{4}{3}k$ and $1.5\log(\log(k))+4.2 \leq 4\log(k)$ for $k \geq 3$, it is sufficient to have

$$\frac{200}{3}k^2\log(k) < 2^D$$

or

$$D = \left\lceil \frac{2\log(k)+\log(\log(k))+4.2}{\log(2)} \right\rceil \leq \frac{2\log(k)+\log(\log(k))+4.2}{\log(2)} + 1. \tag{5.70}$$

Hence, we conclude that

$$|I_2| \leq \left[2^{\frac{3}{2}k^2+\frac{11}{2}k+1+D}k^{\frac{1}{2}k^2+\frac{25}{6}k-2+D}\mathcal{M}_0\right]^{\frac{33}{10}k^{D+1}} \cdot (2k)^{2k^3+11k^2}\cdot X^{2s-\frac{1}{2}k(k+1)-\delta} \tag{5.71}$$

for some $\delta > 0$ and where D is given by (5.70) and $\mathcal{M}_0$ as defined in (5.11). The rest of the calculation goes through as in [ACK04] except that one has to increase the constant to four times the maximum out of k^{30k^3} and the constant in Equation (5.71). Hence, we conclude the following theorem.

Theorem 5.3.22. *Let $k \geq 3$, $s \geq 5k^2+2k$. Furthermore, let $X \geq s^{10}$. We have the asymptotic formula:*

$$\left|I_{s,k,l}(\boldsymbol{N};X) - \mathfrak{S}_{s,k,l}(\boldsymbol{N})\mathcal{J}_{s,k,l}(\boldsymbol{N})X^{s-\frac{1}{2}k(k+1)}\right| \leq C\cdot X^{s-\frac{1}{2}k(k+1)-\delta},$$

as well as the estimate

$$I_{s,k,l}(\boldsymbol{N};X) \leq CX^{s-\frac{1}{2}k(k+1)},$$

where C *is the maximum of* $4k^{30k^3}$ *and*

$$\left[2^{\frac{3}{2}k^2+\frac{11}{2}k+1+D}k^{\frac{1}{2}k^2+\frac{25}{6}k-2+D}\mathcal{M}_0\right]^{\frac{33}{10}k^{D+1}} \cdot 4(2k)^{2k^3+11k^2},$$

where $\mathcal{M}_0$ *as defined in* (5.11) *and*

$$D = \left\lceil \frac{2\log(k) + \log(\log(k)) + 4.2}{\log(2)} \right\rceil.$$

6

THE COVERING EXPONENT OF S^3

6.1 INTRODUCTION

The question about a covering exponent is closely linked to the question of intrinsic Diophantine approximation. To elucidate this, let us review the classical question about Diophantine approximation of a real number ξ. The approximation exponent $\mu(\xi)$ of ξ is defined as the supremum of all real numbers μ such that the set

$$\{(p,q) \in \mathbb{Z} \times \mathbb{N} \| \xi - \tfrac{p}{q}| < q^{-\mu}\}$$

is infinite[a]. We may define a second exponent $\hat{\mu}(\xi)$, which will tell us more about how sparse the above set is. The exponent $\hat{\mu}(\xi)$ is defined as the supremum of all real numbers such that for every sufficiently large Q the set

$$\{(p,q) \in \mathbb{Z} \times \mathbb{N} | q \leq Q \wedge |\xi - \tfrac{p}{q}| < Q^{-\mu}\}$$

is non-empty. Clearly, we have the inequality $\hat{\mu}(\xi) \leq \mu(\xi)$. We may take this one step further and ask about a uniform exponent $\hat{\mu}$ which we define as the supremum of the set

$$\{\mu \in \mathbb{R} | \exists N \in \mathbb{N} : \forall Q \geq N : \forall \xi \in [0,1] : \exists (p,q) \in \mathbb{Z} \times \mathbb{N} : q \leq Q \wedge |\xi - \tfrac{p}{q}| < Q^{-\mu}\}.$$

Equivalently, it is the supremum of all numbers μ such that for large enough Q we have

$$\bigcup_{\substack{(p,q) \in \mathbb{Z} \times \mathbb{N} \\ q \leq Q}} B(\tfrac{p}{q}, Q^{-\mu}) \supseteq [0,1],$$

here $B(x,r)$ denotes the open ball of radius r around x. The covering exponent $K([0,1], \mathbb{Q})$ now represents a normalised version of $\hat{\mu}$ as it shall take into account how many balls we had to use. In this case, the number of balls is of the order Q^2 and the covering exponent is defined as $2/\hat{\mu}$ and we easily find $K([0,1], \mathbb{Q}) = 2$.

a Usually one forces the distance to be positive, but for the sake of comparison we allow 0.

In the general setting, we are given an oriented compact smooth Riemannian manifold $\mathcal{M}$, which usually is given by a compact subset of all real solutions to a set of polynomial equations, and a dense subset $\mathcal{Q}$ together with a height function $H : \mathcal{Q} \to \mathbb{R}_0^+$ such that the sets $H^{-1}([0, Q])$ are all finite for every $Q \in \mathbb{R}_0^+$. For example, $\mathcal{Q}$ might be taken to be the set of all rational points (or S-integers) on $\mathcal{M}$ and H the usual height function. Set $V(Q)$ to be supremum of the volume of all balls $B(\xi, r) \subseteq \mathcal{M}$ which do not contain any point of $H^{-1}([0, Q])$. Then, the covering exponent is defined as

$$K(\mathcal{M}, \mathcal{Q}, H) = \limsup_{Q\to\infty} \frac{\log |H^{-1}([0, Q])|}{\log \operatorname{vol}(\mathcal{M})/V(Q)}. \tag{6.1}$$

The covering exponents satisfy trivially $K(\mathcal{M}, \mathcal{Q}, H) \geq 1$ and measure on an exponential level how far the set $\mathcal{Q}$ is from perfect equidistribution.

The work on covering exponents gained a lot of popularity after Sarnak, in a letter addressed to Aaronson and Pollington [Sar15b], pointed out the connection between the covering exponent of S^3 and efficient quantum computing on 1-qubits via the isomorphism $S^3 \cong \mathrm{SU}_2$. In the same letter, he mentions that a result of Kleinbock–Merrill [KM15] implies $K(S^n, S^n(\mathbb{Q}), H) = 2$ for $n \geq 1$, where H is the usual height function. If one wishes to get a smaller covering exponent one needs to consider sparser subsets of the rationals. Sarnak considers the set of $\{\infty, 5\}$-integers and shows amongst other things that

$$K(S^2, S^2(\mathbb{Z}[\tfrac{1}{5}]), H) \leq 2,$$
$$\tfrac{4}{3} \leq K(S^3, S^3(\mathbb{Z}[\tfrac{1}{5}]), H) \leq 2.$$

Here, the lower bound $\frac{4}{3}$ comes from a Diophantine repulsion property, which forces a large annulus around $(0, 0, 0, 1)$ with no solutions of small denominator. Heuristically, the rational points are the worst approximable numbers and hence Sarnak goes on and conjectures that indeed $K(S^3, S^3(\mathbb{Z}[\frac{1}{5}]), H) = \frac{4}{3}$.

Sardari [Sar15a] takes this one step further and considers only those rational points with a given denominator. In the case of S^n, this clearly constitutes a finite set and hence can't be dense. Therefore, the definition (6.1) needs some tweaking. Let us assume that $\mathcal{M}$ is given by the 1-level set of a single homogeneous polynomial of degree d with integer coefficients. Let $\mathcal{M}_N = N^{\frac{1}{d}} \cdot \mathcal{M}$ denote the N-level set, $\mathcal{M}_N(\mathbb{Z})$ the set of points of $\mathcal{M}_N$ with integer coordinates, and $V(N)$ the volume of the largest ball $B(\xi, r) \subseteq \mathcal{M}_N$

that does not contain a point of $\mathcal{M}_N(\mathbb{Z})$. Then, the integer covering exponent is defined as

$$K(\mathcal{M}, \mathbb{Z}, \mathcal{B}) = \limsup_{\mathcal{B} \ni N \to \infty} \frac{\log |\mathcal{M}_N(\mathbb{Z})|}{\log \operatorname{vol}(\mathcal{M}_N)/V(N)}, \tag{6.2}$$

here $\mathcal{B} \subseteq \mathbb{N}$ is a set that avoids certain "bad" integers. Say for example $\mathcal{B} = 2\mathbb{N} - 1$ in the case of S^3. In his paper, Sardari [Sar15a] manages to prove (amongst other things)

$$K(S^n, \mathbb{Z}, \mathbb{N}) = 2 - \tfrac{2}{n}, \quad \forall n \geq 4,$$

and

$$K(S^3, \mathbb{Z}, 2\mathbb{N} - 1) \leq 2. \tag{6.3}$$

For the sake of completeness, we shall also state a result of Duke–Schulze-Pillot [DSP90], which implies

$$K(S^2, \mathbb{Z}, 4\mathbb{N} + 1) \leq \frac{203}{4}.$$

This follows [DSP90, Lemmata 3 and 5] with the choice of a bump function that approximates a ball of radius δ. We should remark that such a function satisfies condition (iii) of [DSP90, Lemma 1] and $P_0 \gg \delta^2$.

In section 6.3, we reproduce Sardari's result and extend it to show that the twisted Linnik conjecture 4.0.2 implies $K(S^3, \mathbb{Z}, 2\mathbb{N} - 1) = \frac{4}{3}$, from which Sarnak's conjecture $K(S^3, S^3(\mathbb{Z}[\frac{1}{5}]), H) = \frac{4}{3}$ follows. In section 6.2, we shall give a short proof of $K(S^3, \mathbb{Z}, 2\mathbb{N} - 1) \leq \frac{7}{3}$ based on the theory of automorphic forms, followed up by Sarnak's argument for $K(S^3, \mathbb{Z}[\frac{1}{5}]) \leq 2$, which also shows $K(S^3, \mathbb{Z}, 2\mathbb{N} - 1) \leq 2$. In section 6.4, we shall compare the two approaches and discuss how they may be unified.

6.2 AN AUTOMORPHIC APPROACH

Let $\omega : \mathbb{R}_0^+ \to [0, 1]$ be a smooth bump function with $\operatorname{Supp} \omega \subseteq [0, 1]$ and $\omega([0, \frac{1}{2}]) = \{1\}$. Further, let $\boldsymbol{\xi} \in S^3$ be a point and $\epsilon > 0$ a parameter. Our aim is to prove

$$\sum_{\substack{\boldsymbol{x} \in \mathbb{Z}^4 \\ \|\boldsymbol{x}\|_2^2 = N}} \omega\left(\frac{\|\boldsymbol{x}/\sqrt{N} - \boldsymbol{\xi}\|_2}{\epsilon}\right) > 0 \tag{6.4}$$

for an ϵ as small as possible. We note here that $\|\boldsymbol{x} - \boldsymbol{\xi}\|_2^2 = 2(1 - \langle \boldsymbol{x}, \boldsymbol{\xi} \rangle)$ as a function in $\boldsymbol{x} \in S^3$ depends only on the angle $\cos(\theta) = \langle \boldsymbol{x}, \boldsymbol{\xi} \rangle$. Therefore, it makes sense to expand in terms of spherical polynomials, where we chose $\boldsymbol{\xi}$ as the north pole. The spectral expansion (see [Iwa97, Chapter 9]) reads

$$\omega\left(\frac{\|\boldsymbol{x}-\boldsymbol{\xi}\|_2}{\epsilon}\right) = \sum_{n=0}^{\infty} \omega_n U_n(\langle \boldsymbol{x}, \boldsymbol{\xi}\rangle), \tag{6.5}$$

here

$$U_n(\cos(\theta)) = \frac{\sin((n+1)\theta)}{\sin(\theta)}$$

are the Chebyshev polynomials of the second kind and

$$\begin{aligned} \omega_n &= \int_{S^3} \omega\left(\frac{\|\boldsymbol{x}-\boldsymbol{\xi}\|_2}{\epsilon}\right) U_n(\langle \boldsymbol{x}, \boldsymbol{\xi}\rangle) d_{S^3}\boldsymbol{x} \\ &= \frac{2}{\pi}\int_0^{\pi} \omega\left(\frac{\sqrt{2}}{\epsilon}\sqrt{1-\cos(\theta)}\right)\sin(\theta)^2 U_n(\cos(\theta))d\theta \\ &= \frac{2}{\pi}\int_{-1}^{1} \omega\left(\frac{\sqrt{2}}{\epsilon}\sqrt{1-x}\right) U_n(x)\sqrt{1-x^2}dx. \end{aligned} \tag{6.6}$$

We shall list a few properties of the Chebyshev polynomials of the second kind, which we shall need. They are the orthogonal polynomials with respect to the inner product

$$\langle f, g\rangle_U = \frac{2}{\pi}\int_{-1}^{1} f(x)g(x)\sqrt{1-x^2}dx,$$

which is what we secretly exploited to get the expansion (6.5) (as well as the Stone–Weierstrass Theorem). Moreover, we have

$$\Delta_{S^3} U_n = -n(n+2)U_n, \tag{6.7}$$

$$U_n(-x) = (-1)^n U_n(x), \tag{6.8}$$

$$|U_n(\cos(\theta))| \leq \min\{n+1, |\sin(\theta))|^{-1}\}, \tag{6.9}$$

where Δ_{S^3} denotes the Laplace–Beltrami operator on S^3.

Returning to (6.4), we find

$$\sum_{n=0}^{\infty} \omega_n \sum_{\substack{\boldsymbol{x}\in\mathbb{Z}^4 \\ \|\boldsymbol{x}\|_2^2 = N}} U_n\left(\left\langle \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|_2}, \boldsymbol{\xi}\right\rangle\right) \tag{6.10}$$

by inserting the spectral expansion (6.5). By pairing up $\boldsymbol{x}$ with $-\boldsymbol{x}$, we find using (6.8) that the terms with n odd vanish. The term with $n = 0$ will be our main term. By (6.6) and $U_0 \equiv 1$, we have

$$\omega_0 \sum_{\substack{\boldsymbol{x}\in\mathbb{Z}^4 \\ \|\boldsymbol{x}\|_2^2 = N}} U_0\left(\left\langle \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|_2}, \boldsymbol{\xi}\right\rangle\right) = \omega_0 r_4(N) \gg \epsilon^3 r_4(N). \tag{6.11}$$

It remains to analyse the terms with $n > 0$ even. We start by bounding ω_n.

Lemma 6.2.1. *Let $n > 0$ and $A \in \mathbb{N}_0$. Then, we have the bound*

$$\omega_n \ll_A \epsilon^2 \min\{1, \epsilon n\}(\epsilon n)^{-2A}$$

Proof. We have

$$\begin{aligned}
(-n(n+2))^A \omega_n &= \int_{S^3} \omega\left(\frac{\|\boldsymbol{x}-\boldsymbol{\xi}\|_2}{\epsilon}\right) \Delta_{S^3}^A U_n(\langle \boldsymbol{x}, \boldsymbol{\xi}\rangle) d_{S^3}\boldsymbol{x} \\
&= \int_{S^3} \Delta_{S^3}^A \omega\left(\frac{\|\boldsymbol{x}-\boldsymbol{\xi}\|_2}{\epsilon}\right) U_n(\langle \boldsymbol{x}, \boldsymbol{\xi}\rangle) d_{S^3}\boldsymbol{x} \\
&\leq \frac{2}{\pi}\int_0^\pi \left|\Delta_{S^3}^A \omega\left(\frac{\sqrt{2}}{\epsilon}\sqrt{1-\cos(\theta)}\right)\right| \sin(\theta)^2 \min\{n, \sin(\theta)^{-1}\} d\theta \\
&\ll_A \epsilon^{-2A} \int_0^\epsilon \sin(\theta)^2 \min\{n, \sin(\theta)^{-1}\} d\theta \\
&\ll_A \epsilon^2 \min\{1, \epsilon n\} \cdot \epsilon^{-2A},
\end{aligned}$$

where we have made use of the self-adjointness of the Laplace–Beltrami operator and (6.9). □

In order to bound

$$\sum_{\substack{\boldsymbol{x}\in\mathbb{Z}^4 \\ \|\boldsymbol{x}\|_2^2 = N}} U_n\left(\left\langle \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|_2}, \boldsymbol{\xi}\right\rangle\right),$$

we shall relate it to a Fourier coefficient $\widehat{F_n}(N)$ of a function $F_n : \mathbb{H} \to \mathbb{C}$, which we define as

$$\begin{aligned}
F_n(z) &= \sum_{\boldsymbol{x}\in\mathbb{Z}^4} \|\boldsymbol{x}\|_2^n U_n\left(\left\langle \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|_2}, \boldsymbol{\xi}\right\rangle\right) e\left(\frac{1}{2}\|\boldsymbol{x}\|_2^2 z\right) \\
&= \sum_{m=1}^\infty \widehat{F_n}(m) e\left(\frac{1}{2}mz\right).
\end{aligned} \tag{6.12}$$

We shall require a lemma on Fourier transforms.

Lemma 6.2.2. *Let P be a polynomial in d variables. Then, we have*

$$\begin{aligned}
\mathcal{F}\left[P(\boldsymbol{x})e^{-\pi\|\boldsymbol{x}\|_2^2}\right](\boldsymbol{\omega}) &= \int_{\mathbb{R}^d} P(\boldsymbol{x}) e^{-\pi\|\boldsymbol{x}\|_2^2} e^{-2\pi i\langle \boldsymbol{x}, \boldsymbol{\omega}\rangle} d\boldsymbol{x} \\
&= \left[\exp\left(\frac{\Delta_{\mathbb{R}^d}}{4\pi}\right) P\right](-i\boldsymbol{\omega}) \cdot e^{-\pi\|\boldsymbol{\omega}\|_2^2},
\end{aligned}$$

here $\Delta_{\mathbb{R}^d}$ denotes the Laplace–Beltrami operator on $\mathbb{R}^d$.

Proof. By using linearity, it suffices to prove this for monomials. Further, using Fubini it suffices to prove this in the case $d = 1$. We shall prove the claim inductively in the degree

m of P. If $m \leq 0$, then this follows from the well-known equality $\mathcal{F}[e^{-\pi x^2}](\omega) = e^{-\pi\omega^2}$. For the induction step, we may now assume that $P(0) = 0$ and write $P(x) = xQ(x)$. We have

$$\begin{aligned}
\mathcal{F}[xQ(x)e^{-\pi x^2}](\omega) &= -\frac{1}{2\pi i}\frac{d}{d\omega}\mathcal{F}[Q(x)e^{-\pi x^2}](\omega) \\
&= -\frac{1}{2\pi i}\frac{d}{d\omega}\left[\exp\left(\frac{\Delta_{\mathbb{R}}}{4\pi}\right)Q\right](-i\omega)\cdot e^{-\pi\omega^2} \\
&= \frac{1}{2\pi}\left[\frac{d}{dx}\exp\left(\frac{\Delta_{\mathbb{R}}}{4\pi}\right)Q\right](-i\omega)e^{-\pi\omega^2} + \frac{\omega}{i}\left[\exp\left(\frac{\Delta_{\mathbb{R}}}{4\pi}\right)Q\right](-i\omega)e^{-\pi\omega^2} \\
&= \left[x\exp\left(\frac{\Delta_{\mathbb{R}}}{4\pi}\right)Q + \frac{1}{2\pi}\frac{d}{dx}\exp\left(\frac{\Delta_{\mathbb{R}}}{4\pi}\right)Q\right](-i\omega)e^{-\pi\omega^2} \\
&= \left[\exp\left(\frac{\Delta_{\mathbb{R}}}{4\pi}\right)P\right](-i\omega)e^{-\pi\omega^2}.
\end{aligned}$$

□

As a Corollary, we immediately find that

$$\mathcal{F}\left[\|\boldsymbol{x}\|_2^n U_n\left(\left\langle\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|_2},\boldsymbol{\xi}\right\rangle\right)e^{-\pi\|\boldsymbol{x}\|_2^2}\right](\omega) = (-i)^n\|\boldsymbol{\omega}\|_2^n U_n\left(\left\langle\frac{\boldsymbol{\omega}}{\|\boldsymbol{\omega}\|_2},\boldsymbol{\xi}\right\rangle\right)e^{-\pi\|\boldsymbol{\omega}\|_2^2}, \quad (6.13)$$

since

$$\Delta_{S^3}U_n(\langle\boldsymbol{x},\boldsymbol{\xi}\rangle) = -n(n+2)U_n(\langle\boldsymbol{x},\boldsymbol{\xi}\rangle) \Leftrightarrow \Delta_{\mathbb{R}^4}\|\boldsymbol{x}\|_2^n U_n\left(\left\langle\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|_2},\boldsymbol{\xi}\right\rangle\right) = 0.$$

Poisson summation now tells us that for $\boldsymbol{\lambda} \in \mathbb{R}^4$ and $t > 0$ we have

$$\begin{aligned}
&\sum_{\boldsymbol{x}\in\mathbb{Z}^4}\|\boldsymbol{x}+\boldsymbol{\lambda}\|_2^n U_n\left(\left\langle\frac{\boldsymbol{x}+\boldsymbol{\lambda}}{\|\boldsymbol{x}+\boldsymbol{\lambda}\|_2},\boldsymbol{\xi}\right\rangle\right)e^{-\pi t\|\boldsymbol{x}+\boldsymbol{\lambda}\|_2^2} \\
&\qquad = (-i)^n t^{-n-2}\sum_{\boldsymbol{\omega}\in\mathbb{Z}^4}\|\boldsymbol{\omega}\|_2^n U_n\left(\left\langle\frac{\boldsymbol{\omega}}{\|\boldsymbol{\omega}\|_2},\boldsymbol{\xi}\right\rangle\right)e^{-\pi\frac{1}{t}\|\boldsymbol{\omega}\|_2^2}e^{2\pi i\langle\boldsymbol{\omega},\boldsymbol{\lambda}\rangle}. \quad (6.14)
\end{aligned}$$

Two values of $\boldsymbol{\lambda}$ are of importance to us, namely $\boldsymbol{\lambda} = \boldsymbol{0}$ and $\boldsymbol{\lambda} = \frac{\boldsymbol{1}}{\boldsymbol{2}}$. By analytic continuation in the variable it, the former yields

$$F_n(z) = -z^{-n-2}F_n\left(-\frac{1}{z}\right), \quad \forall z \in \mathbb{H}, \quad (6.15)$$

or simply $F_n|_{n+2}\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right) = -F_n$, and the latter yields

$$z^{-n-2}F_n\left(1-\frac{1}{z}\right) = -\sum_{\boldsymbol{x}\in\mathbb{Z}^4+\frac{\boldsymbol{1}}{\boldsymbol{2}}}\|\boldsymbol{x}\|_2^n U_n\left(\left\langle\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|_2},\boldsymbol{\xi}\right\rangle\right)e\left(\frac{1}{2}\|\boldsymbol{x}\|_2^2 z\right), \quad \forall z \in \mathbb{H}. \quad (6.16)$$

Equation (6.15) together with $F_n(z+2) = F_n(z) \Leftrightarrow F_n|_{n+2}\left(\begin{smallmatrix}1 & 2\\ 0 & 1\end{smallmatrix}\right) = F_n$, which can be easily seen from the definition (6.12), implies that F_n is modular of weight $n+2$ with respect to the theta subgroup Γ_θ, as Γ_θ is generated by the matrices $-I$, $\left(\begin{smallmatrix}1 & 2\\ 0 & 1\end{smallmatrix}\right)$, and $\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$.

The theta subgroup has two cusp ∞ and 1. Hence, the expansions (6.12) and (6.16) show that F_n is a cusp form (recall $n > 0$).

We now intend to expand F_n in terms of an orthonormal basis of Hecke eigenforms. To this end, we note that $G_n(z) = 2^{\frac{n+2}{2}} F_n(2z) = (F|_{n+2}A_2)(z)$ is a cusp form of weight $n+2$ with respect to $\Gamma_0(4)$ and trivial character. Since $\Gamma_0(4)$ is generated by $-I, \left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1 & 0\\ 4 & 1\end{smallmatrix}\right)$, this follows from the two matrix identities

$$\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}^{-1} = \begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix} \in \Gamma_\theta$$

and

$$\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 4 & 1\end{pmatrix}\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}^{-1} = \begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix} = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}^{-1}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}^{-1} \in \Gamma_\theta.$$

We shall require a bound on the Petersson norm of H_n.

Proposition 6.2.3. *Let $n > 0$ be an even integer. Then, we have*

$$\|G_n\|^2 = \int_{\mathcal{F}_{\Gamma_0(4)}} |G_n(z)|^2 y^{n+2} \frac{dxdy}{y^2} \ll_\epsilon (2\pi)^{-n}\Gamma(n+2)\cdot n^{2+\epsilon}.$$

Proof. We have

$$\begin{aligned}\int_{\mathcal{F}_{\Gamma_0(4)}} |G_n(z)|^2 y^{n+2} \frac{dxdy}{y^2} &= \int_{\mathcal{F}_{\Gamma_0(4)}} |F_n(2z)|^2 (2y)^{n+2} \frac{dxdy}{y^2}\\ &= \int_{\mathcal{F}_{\Gamma(2)}} |F_n(z)|^2 y^{n+2} \frac{dxdy}{y^2}\\ &= 2\int_{\mathcal{F}_{\Gamma_\theta}} |F_n(z)|^2 y^{n+2} \frac{dxdy}{y^2}.\end{aligned}$$

We further bound the latter by

$$2\left(\int_{\frac{\sqrt{3}}{2}}^{\infty}\int_0^2 |F_n(z)|^2 y^n dxdy + \int_{\frac{\sqrt{3}}{2}}^{\infty}\int_0^1 \left|z^{-n-2}F_n\left(1-\frac{1}{z}\right)\right|^2 y^n dxdy\right) = 2(\mathcal{I}_1 + \mathcal{I}_2), \text{ say.}$$

We shall only deal with $\mathcal{I}_1$ as $\mathcal{I}_2$ may be treated in the same fashion. We insert the Fourier expansion (6.12) and integrate over x. We find

$$\mathcal{I}_1 = \int_{\frac{\sqrt{3}}{2}}^{\infty} \sum_{\substack{\boldsymbol{m},\boldsymbol{l} \in \mathbb{Z}^4 \\ \|\boldsymbol{m}\|_2^2 = \|\boldsymbol{l}\|_2^2}} \|\boldsymbol{m}\|_2^n U_n\left(\left\langle \frac{\boldsymbol{m}}{\|\boldsymbol{m}\|_2}, \boldsymbol{\xi} \right\rangle\right) \|\boldsymbol{l}\|_2^n U_n\left(\left\langle \frac{\boldsymbol{l}}{\|\boldsymbol{l}\|_2}, \boldsymbol{\xi} \right\rangle\right) e^{-\pi y(\|\boldsymbol{m}\|_2^2 + \|\boldsymbol{l}\|_2^2)} y^n dy$$

$$= \int_{\frac{\sqrt{3}}{2}}^{\infty} \sum_{k=1}^{\infty} k^n e^{-2\pi k y} \left(\sum_{\substack{\boldsymbol{m} \in \mathbb{Z}^4 \\ \|\boldsymbol{m}\|_2^2 = k}} U_n\left(\left\langle \frac{\boldsymbol{m}}{\|\boldsymbol{m}\|_2}, \boldsymbol{\xi} \right\rangle\right) \right)^2 y^n dy$$

$$\leq \int_{\frac{\sqrt{3}}{2}}^{\infty} \sum_{k=1}^{\infty} k^n e^{-2\pi k y} \left(\sum_{\substack{\boldsymbol{m} \in \mathbb{Z}^4 \\ \|\boldsymbol{m}\|_2^2 = k}} \min\left\{ n+1, \frac{\|\boldsymbol{m}\|_2}{\sqrt{\|\boldsymbol{m}\|_2^2 - \langle \boldsymbol{m}, \boldsymbol{\xi} \rangle^2}} \right\} \right)^2 y^n dy.$$

Let us first deal with the part where $k \geq 10n$. In this case, we have that the inner sum is bounded by

$$n^2 \sum_{k \geq 10n} k^{n+3} e^{-2\pi k y} \ll n^2 \sum_{k \geq 10n} n^{n+3} (2\pi y)^{-n-3} e^{-n} e^{-\pi k y}$$
$$\ll n^{n+5} (2\pi e)^{-n} y^{-n-3} e^{-10\pi n y}.$$

Hence, the contribution from $k \geq 10n$ towards $\mathcal{I}_1$ is bounded by

$$n^{n+5} (2\pi e)^{-n} \int_{\frac{\sqrt{3}}{2}}^{\infty} e^{-10\pi n y} y^{-3} dy \ll n^{n+5} (2\pi e)^{-n} e^{-10n}. \tag{6.17}$$

This is sufficient. For $k \leq 10n$, we interchange the integral and summation in $\mathcal{I}_1$. We further extend the integral all the way down to 0 and find that the contribution is at most

$$2(2\pi)^{1-n}\Gamma(n) \sum_{k=1}^{10n} k \left(\sum_{\substack{\boldsymbol{m} \in \mathbb{Z}^4 \\ \|\boldsymbol{m}\|_2^2 = k}} \min\left\{ n, \frac{\|\boldsymbol{m}\|_2}{\sqrt{\|\boldsymbol{m}\|_2^2 - \langle \boldsymbol{m}, \boldsymbol{\xi} \rangle^2}} \right\} \right)^2$$
$$= 2(2\pi)^{1-n}\Gamma(n) \left(10nA(10n) - \sum_{k=1}^{10n-1} A(k) \right), \tag{6.18}$$

where

$$A(X) = \sum_{k \leq X} \left(\sum_{\substack{\boldsymbol{m} \in \mathbb{Z}^4 \\ \|\boldsymbol{m}\|_2^2 = k}} \min\left\{ n, \frac{\|\boldsymbol{m}\|_2}{\sqrt{\|\boldsymbol{m}\|_2^2 - \langle \boldsymbol{m}, \boldsymbol{\xi} \rangle^2}} \right\} \right)^2.$$

It suffices to bound $A(X)$ from above. For this matter, we need to borrow two propositions from the geometry of numbers.

Proposition 6.2.4 (Minkowski's second Theorem)**.** *Let $\mathcal{K} \subseteq \mathbb{R}^n$ be a closed convex centrally symmetric set of positive volume. Let $\Lambda \subset \mathbb{R}^n$ be a lattice and further let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the successive minima of $\mathcal{K}$ on Λ. Then, we have*

$$\frac{2^n}{n!}\operatorname{vol}(\mathbb{R}^n/\Lambda) \leq \lambda_1\lambda_2\cdots\lambda_n \operatorname{vol}(\mathcal{K}) \leq 2^n \operatorname{vol}(\mathbb{R}^n/\Lambda).$$

Proposition 6.2.5. *Let $\mathcal{K} \subseteq \mathbb{R}^n$ be a closed convex centrally symmetric set of positive volume. Let $\Lambda \subset \mathbb{R}^n$ be a lattice and further let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the successive minima of $\mathcal{K}$ on Λ. Then, we have*

$$|\mathcal{K} \cap \Lambda| \leq \prod_{i=1}^{n}\left(1 + \frac{2i}{\lambda_i}\right).$$

Proof. See [BHW93, Prop. 2.1]. □

In order to bound $A(X)$, we partition the points $\boldsymbol{m}$ into sets of the type $B(R)$, which are defined as follows

$$\boldsymbol{m} \in B(R) \Leftrightarrow R \leq \frac{\|\boldsymbol{m}\|_2}{\sqrt{\|\boldsymbol{m}\|_2^2 - \langle \boldsymbol{m}, \boldsymbol{\xi}\rangle^2}} \leq 2R. \tag{6.19}$$

We shall make a change of variables. Extend $\boldsymbol{e_1} = \boldsymbol{\xi}$ to an orthonormal basis $\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3}, \boldsymbol{e_4}$ of $\mathbb{R}^4$ and set $\hat{m}_i = \langle \boldsymbol{m}, \boldsymbol{e_i}\rangle$. Then, we see that the condition (6.19) implies

$$\hat{m}_2^2 + \hat{m}_3^2 + \hat{m}_4^2 \leq \frac{\|\boldsymbol{m}\|_2^2}{R^2}. \tag{6.20}$$

Fix a k and consider all points $\boldsymbol{m} \in B(R)$ with $\|\boldsymbol{m}\|_2^2 = k$. They are 1-separated. Further note, that the cylinder (6.20) intersects each ball of radius $\frac{1}{2}$ around a point $\boldsymbol{m} \in B(R)$ with $\|\boldsymbol{m}\|_2^2 = k$ with a (uniform) proportion of its volume. Hence, we deduce

$$\left|\left\{\boldsymbol{m} \in \mathbb{Z}^4 | \|\boldsymbol{m}\|_2^2 = k \text{ and } \boldsymbol{m} \in B(R)\right\}\right| \ll \min\left\{k^{1+o(1)}, 1 + \frac{k^{\frac{3}{2}}}{R^3}\right\}. \tag{6.21}$$

We are now going to refine this estimate as k varies in an interval $[M, 2M]$. In this case, we have the conditions

$$\hat{m}_1^2 \leq 2M \text{ and } \hat{m}_2^2 + \hat{m}_3^2 + \hat{m}_4^2 \leq \frac{\|\boldsymbol{m}\|_2^2}{R^2}. \tag{6.22}$$

This defines a centrally symmetric cylinder $\mathcal{K}$. By Proposition 6.2.5, the number of points $\boldsymbol{m}$ inside $\mathcal{K}$ is bounded by

$$\ll \frac{1}{\lambda_1} + \frac{1}{\lambda_1\lambda_2} + \frac{1}{\lambda_1\lambda_2\lambda_3} + \frac{1}{\lambda_1\lambda_2\lambda_3\lambda_4}.$$

Clearly, we have $\lambda_1 \gg M^{-\frac{1}{2}}$ and $\lambda_1\lambda_2\lambda_3\lambda_4 \gg M^{-2}R^3$ by Proposition 6.2.4. We also claim $\lambda_1\lambda_2 \gg M^{-1}R$ and $\lambda_1\lambda_2\lambda_3 \gg M^{-\frac{3}{2}}R^2$. Let us illustrate this for $\lambda_1\lambda_2$. Let $\boldsymbol{v_1}, \boldsymbol{v_2}$ be two linearly independent vectors for which the second successive minima is attained. Then, $\mathbb{Z}\boldsymbol{v_1} + \mathbb{Z}\boldsymbol{v_2}$ is a lattice with co-volume at least 1 and $\mathrm{vol}(\mathcal{K} \cap (\mathbb{R}\boldsymbol{v_1} + \mathbb{R}\boldsymbol{v_2})) \ll MR^{-1}$ and hence by Proposition 6.2.4 we have $\lambda_1\lambda_2 \gg M^{-1}R$. The bound $\lambda_1\lambda_2\lambda_3 \gg M^{-\frac{3}{2}}R^2$ follows from the same considerations. Thus, we find

$$\left|\left\{\boldsymbol{m} \in \mathbb{Z}^4 | M \leq \|\boldsymbol{m}\|_2^2 \leq 2M \text{ and } \boldsymbol{m} \in B(R)\right\}\right| \ll M^{\frac{1}{2}} + \frac{M^2}{R^3}. \tag{6.23}$$

We shall remark here that the bounds (6.21) and (6.23) still hold if we replace the set $B(R)$ with the set $C(R)$, where

$$\boldsymbol{m} \in C(R) \Leftrightarrow R \leq \frac{\|\boldsymbol{m}\|_2}{\sqrt{\|\boldsymbol{m}\|_2^2 - \langle \boldsymbol{m}, \boldsymbol{\xi}\rangle^2}}.$$

We shall make use of this when $R \geq n$. From Cauchy–Schwarz, it follows that

$$\begin{aligned} A(2M) - A(M) &= \sum_{M \leq k \leq 2M} \left(\sum_{\substack{\boldsymbol{m} \in \mathbb{Z}^4 \\ \|\boldsymbol{m}\|_2^2 = k}} \min\left\{ n, \frac{\|\boldsymbol{m}\|_2}{\sqrt{\|\boldsymbol{m}\|_2^2 - \langle \boldsymbol{m}, \boldsymbol{\xi}\rangle^2}} \right\} \right)^2 \\ &\ll \sum_{M \leq k \leq 2M} \left(\sum_{i=0}^{\lfloor \log_2(n) \rfloor} \mu_i + \mu \right) \left(\sum_{i=0}^{\lfloor \log_2(n) \rfloor} \frac{2^{2i}}{\mu_i} \left(\sum_{\substack{\|\boldsymbol{m}\|_2^2 = k \\ \boldsymbol{m} \in B(2^i)}} 1 \right)^2 + \frac{n^2}{\mu} \left(\sum_{\substack{\|\boldsymbol{m}\|_2^2 = k \\ \boldsymbol{m} \in C(n)}} 1 \right)^2 \right) \end{aligned} \tag{6.24}$$

for some positive weights μ_i, μ, which we shall choose in due time. Equations (6.21) and (6.23) imply

$$\sum_{M \leq k \leq 2M} \left(\sum_{\substack{\|\boldsymbol{m}\|_2^2 = k \\ \boldsymbol{m} \in B(R)}} 1 \right)^2 \ll \min\left\{ M^{\frac{1}{2}} + \frac{M^{\frac{7}{2}}}{R^6}, M^{\frac{3}{2}+o(1)} + \frac{M^{3+o(1)}}{R^3} \right\}.$$

Hence, (6.24) is further bounded by

$$\begin{aligned} &\left(\sum_{i=0}^{\lfloor \log_2(n) \rfloor} \mu_i + \mu \right) \left(\sum_{i=0}^{\lfloor \frac{1}{6}\log_2(M) \rfloor} \frac{2^{2i}}{\mu_i} \frac{M^{3+o(1)}}{2^{3i}} + \sum_{i=\lfloor \frac{1}{6}\log_2(M) \rfloor + 1}^{\lfloor \log_2(n) \rfloor} \frac{2^{2i}}{\mu_i} \left(M^{\frac{1}{2}} + \frac{M^{\frac{7}{2}}}{2^{6i}} \right) \right) \\ &\qquad + \left(\sum_{i=0}^{\lfloor \log_2(n) \rfloor} \mu_i + \mu \right) \frac{n^2}{\mu} \left(M^{\frac{1}{2}} + \frac{M^{\frac{7}{2}}}{n^6} \right). \end{aligned} \tag{6.25}$$

We make the following choices for the weights: $\mu = n \cdot M^{\frac{1}{4}}$ and

$$\mu_i = \begin{cases} M^{\frac{3}{2}} 2^{-\frac{1}{2}i}, & 0 \leq i \leq \frac{1}{6}\log_2(M), \\ M^{\frac{7}{4}} 2^{-2i}, & \frac{1}{6}\log_2(M) < i \leq \frac{1}{2}\log_2(M), \\ M^{\frac{1}{4}} 2^{i}, & \frac{1}{2}\log_2(M) < i \leq \lfloor \log_2(n) \rfloor. \end{cases}$$

We find that for $M \ll n$ we have

$$A(2M) - A(M) \ll M^{3+o(1)} + n^2 M^{\frac{1}{2}}$$

and hence $A(10n) \ll n^{3+o(1)}$, from which the Proposition follows.

□

Corollary 6.2.6. *Let $N > 0$ be an odd integer and $n > 0$ be an even integer. Then, we have*

$$|\widehat{F}_n(N)| \ll n^{\frac{3}{2}+o(1)} N^{\frac{n+1}{2}+o(1)}.$$

Proof. Let $\mathcal{B}_{n+2}$ be the Hecke basis (3.52) of weight $n+2$ for $\Gamma_0(4)$. We have

$$\begin{aligned} \widehat{F}_n(N) &= 2^{-\frac{n}{2}-1} \widehat{G}_n(N) \\ &= 2^{-\frac{n}{2}-1} \sum_{f \in \mathcal{B}_{n+2}} \langle G_n, f \rangle \widehat{f}(N) \\ &\ll \frac{(2\pi)^{\frac{n}{2}}}{\Gamma(n+2)^{\frac{1}{2}}} N^{\frac{n+1}{2}+o(1)} \sum_{f \in \mathcal{B}_{n+2}} |\langle G_n, f \rangle| \\ &\ll \frac{(2\pi)^{\frac{n}{2}}}{\Gamma(n+2)^{\frac{1}{2}}} N^{\frac{n+1}{2}+o(1)} |\mathcal{B}_{n+2}|^{\frac{1}{2}} \|G_n\| \\ &\ll n^{\frac{3}{2}+o(1)} N^{\frac{n+1}{2}+o(1)}. \end{aligned}$$

□

Combining everything, that is (6.10), (6.11), and Lemma 6.2.1, we find

$$\sum_{\substack{\boldsymbol{x} \in \mathbb{Z}^4 \\ \|\boldsymbol{x}\|_2^2 = N}} \omega\left(\frac{\|\boldsymbol{x}/\sqrt{N} - \boldsymbol{\xi}\|_2}{\epsilon}\right) = \epsilon^3 r_4(N) + O\left(\epsilon^{-\frac{1}{2}+o(1)} N^{\frac{1}{2}+o(1)}\right).$$

In particular, we find

$$\sum_{\substack{\boldsymbol{x} \in \mathbb{Z}^4 \\ \|\boldsymbol{x}\|_2^2 = N}} \omega\left(\frac{\|\boldsymbol{x}/\sqrt{N} - \boldsymbol{\xi}\|_2}{\epsilon}\right) > 0$$

as soon as $\epsilon \geq N^{-\frac{1}{7}+\delta}$, for a $\delta > 0$. This shows $K(S^3, \mathbb{Z}, 2\mathbb{N}-1) \leq \frac{7}{3}$. This argument is slightly inefficient as we use an L^2-decomposition in two different spaces. It is more efficient to stay in the realm of harmonic polynomials and use the theory of Hecke operators there. This is Sarnak's argument [Sar15b]. In order to describe the argument, we shall identify S^3 with the quaternions of norm 1. Let $Q(\mathbb{Z}) = \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}k$ denote the integer quaternions. As in [LPS87], we can define the Hecke operators on $L^2(S^3)$ for m odd by

$$(T_m f)(\boldsymbol{x}) = \frac{1}{8} \sum_{\substack{\alpha \in Q(\mathbb{Z}) \\ \|\alpha\|_2^2 = m}} f\left(\frac{\alpha}{\|\alpha\|_2} \cdot \boldsymbol{x}\right).$$

They are self-adjoint operators which commute with the Laplace–Beltrami operator Δ_{S^3}. Moreover, they satisfy

$$T_m T_l = T_{ml} = T_l T_m, \quad (m, l) = 1,$$

and

$$T_m T_l = \sum_{d|(m,l)} d \cdot T_{\frac{ml}{d^2}}.$$

Since the $-n(n+2)$-eigenspace of $L^2(S^3)$ is finite-dimensional, we simultaneously diagonalise this space and find an orthonormal basis of eigenfunctions ϕ_j. Let $\lambda_j(m)$ denote the eigenvalue of ϕ_j with respect to T_m. By the same arguments as in [LPS87], we find $|\lambda_j(m)| \ll m^{\frac{1}{2}+o(1)}$ for m odd. We shall briefly sketch this argument. As before we can build a holomorphic cusp form $F(z)$ from $f(\boldsymbol{x})$, a harmonic polynomial of degree n. It turns out that this map commutes with the Hecke operators in the sense that the modular form attached to $(T_m f)(\boldsymbol{x})$ is the same as $m^{1-\frac{k}{2}}(F|_{n+2}T_m)(z)$ for m odd. Hence, the bound for $\lambda_j(m)$ follows from Deligne bound for the eigenvalues of holomorphic newforms.

Moving on, we claim that we have the following expansion

$$U_n(\langle \boldsymbol{x}, \boldsymbol{\xi} \rangle) = \frac{1}{n+1} \sum_j \phi_j(\boldsymbol{\xi}) \phi_j(\boldsymbol{x}). \tag{6.26}$$

We start with the L^2-expansion

$$U_n(\langle \boldsymbol{x}, \boldsymbol{\xi} \rangle) = \sum_j \mu_j(\boldsymbol{\xi}) \phi_j(\boldsymbol{x}),$$

with $\mu_j(\boldsymbol{\xi}) = \langle U_n(\langle \cdot, \boldsymbol{\xi} \rangle), \phi_j \rangle$. We compute this integral using spherical coordinates with $\boldsymbol{\xi}$ being the north pole. By integrating first over the variables different from $\arccos(\langle \boldsymbol{x}, \boldsymbol{\xi} \rangle)$, we see that we leave $U_n(\langle \cdot, \boldsymbol{\xi} \rangle)$ invariant and we just average ϕ_j along those variables.

The resulting function is an eigenfunction of Δ_{S^3} with eigenvalue $-n(n+2)$ that is rotationally symmetric around $\boldsymbol{\xi}$ and thus is a multiple of $U_n(\langle\cdot,\boldsymbol{\xi}\rangle)$. One easily checks that the scaling factor is $\phi_j(\boldsymbol{\xi})/(n+1)$ and hence $\mu_j(\boldsymbol{\xi}) = \phi_j(\boldsymbol{\xi})/(n+1)$. As a consequence we find

$$1 = \|U_n(\langle\cdot,\boldsymbol{\xi}\rangle)\|_2^2 = \sum_j |\mu_j(\boldsymbol{\xi})|^2 = \sum_j \frac{|\phi_j(\boldsymbol{\xi})|^2}{(n+1)^2}, \quad \forall \boldsymbol{\xi} \in S^3.$$

We conclude

$$\begin{aligned}\sum_{\substack{\boldsymbol{x}\in\mathbb{Z}^4\\ \|\boldsymbol{x}\|_2^2=N}} U_n\left(\left\langle \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|_2},\boldsymbol{\xi}\right\rangle\right) &= \frac{1}{n+1}\sum_j (T_N\phi_j)(1)\cdot\phi_j(\boldsymbol{\xi}) \\ &= \frac{1}{n+1}\sum_j \lambda_j(N)\phi_j(1)\phi_j(\boldsymbol{\xi}) \\ &\ll \frac{N^{\frac{1}{2}+o(1)}}{n}\left(\sum_j |\phi_j(1)|^2\right)^{\frac{1}{2}}\left(\sum_j |\phi_j(\boldsymbol{\xi})|^2\right)^{\frac{1}{2}} \\ &\ll n\cdot N^{\frac{1}{2}+o(1)}.\end{aligned}$$

This, in turn, yields

$$\sum_{\substack{\boldsymbol{x}\in\mathbb{Z}^4\\ \|\boldsymbol{x}\|_2^2=N}} \omega\left(\frac{\|\boldsymbol{x}/\sqrt{N}-\boldsymbol{\xi}\|_2}{\epsilon}\right) = \epsilon^3 r_4(N) + O\left(\epsilon^{o(1)}N^{\frac{1}{2}+o(1)}\right)$$

and consequently $K(S^3,\mathbb{Z},2\mathbb{N}-1)\leq 2$.

6.3 A CIRCLE-METHOD APPROACH

In this section, we illustrate a circle-method approach to the problem based on the smooth delta symbol circle method 5.2. Building on the work of Sardari [Sar15a], who used this approach to show $K(S^3,\mathbb{Z},2\mathbb{N}-1)\leq 2$ and $K(S^n,\mathbb{Z},\mathbb{N}) = 2-\frac{2}{n}$ for all $n\geq 4$, we shall establish $K(S^3,\mathbb{Z},8\mathbb{N}+4) = \frac{4}{3}$ under the assumption that the twisted Linnik–Selberg Conjecture on Kloosterman sums 4.0.2 holds. The subsequent work is taken from a collaboration of the author with Browning and Kumaraswamy [BKS17] to which we all contributed equally.

In contrast to the automorphic approach, we shall make use of the ambient space $\mathbb{R}^4$. However, we shall choose a weight function that approximates an ϵ-ball on the sphere S^3. We do this by choosing a weight function which limits the to S^3 normal direction. Let

$w_0 : \mathbb{R} \to \mathbb{R}_{\geq 0}$ be a smooth weight function with unit mass, such that $\operatorname{Supp}(w_0) = [-1, 1]$. We will work with the weight function $w : \mathbb{R}^4 \to \mathbb{R}_{\geq 0}$, given by

$$w(\boldsymbol{x}) = w_0\left(\frac{\|\boldsymbol{x}-\boldsymbol{\xi}\|_2}{\epsilon}\right) w_0\left(\frac{2\boldsymbol{\xi}\cdot(\boldsymbol{x}-\boldsymbol{\xi})}{\epsilon^2}\right). \tag{6.27}$$

Our aim will be to show

$$\Sigma(w) = \sum_{\substack{\boldsymbol{x}\in\mathbb{Z}^4 \\ \|\boldsymbol{x}\|_2^2 = N}} w\left(\frac{\boldsymbol{x}}{\sqrt{N}}\right) > 0,$$

for any $N \in 8\mathbb{N}+4$ large enough. More precisely, we shall prove the following Theorem.

Theorem 6.3.1. *Assume Conjecture 4.0.2. Then, we have*

$$\Sigma(w) = \frac{\epsilon^3 N \sigma_\infty \mathfrak{S}}{2} + O\left(\epsilon^4 N^{1+o(1)} + \epsilon^{\frac{5}{2}} N^{\frac{3}{4}+o(1)} + \epsilon N^{\frac{1}{2}+o(1)}\right),$$

where $\sigma_\infty \gg 1$ is the real density of solutions and $\mathfrak{S} \gg N^{o(1)}$ is the product of non-archimedean local densities, given by

$$\mathfrak{S} = \prod_p \sigma_p, \quad \sigma_p = \lim_{k\to\infty} p^{-3k} \#\{\boldsymbol{x} \in (\mathbb{Z}/p^k\mathbb{Z})^4 : F(\boldsymbol{x}) \equiv N \bmod p^k\}. \tag{6.28}$$

As a consequence one finds $K(S^3, \mathbb{Z}, 8\mathbb{N}+4) = \frac{4}{3}$. We shall show this by making use of the smooth delta symbol circle method 5.2.1. We find

$$\Sigma(w) = \frac{c_Q}{Q^2} \sum_{q=1}^{\infty} \sum_{\boldsymbol{c}\in\mathbb{Z}^4} \frac{1}{q^4} S_q(\boldsymbol{c}) I_q(\boldsymbol{c}), \tag{6.29}$$

where

$$\begin{aligned} S_q(\boldsymbol{c}) &= \sideset{}{^*}\sum_{a \bmod q} \sum_{\boldsymbol{b} \bmod q} e_q\left(a\left(\|\boldsymbol{b}\|_2^2 - N\right) + \boldsymbol{b}\cdot\boldsymbol{c}\right), \\ I_q(\boldsymbol{c}) &= \int_{\mathbb{R}^4} w\left(\frac{\boldsymbol{x}}{\sqrt{N}}\right) h\left(\frac{q}{Q}, \frac{\|\boldsymbol{x}\|_2^2 - N}{Q^2}\right) e_q(-\boldsymbol{c}\cdot\boldsymbol{x})\, d\boldsymbol{x}. \end{aligned} \tag{6.30}$$

Our specific choice of ω will allow us to take $Q = \epsilon\sqrt{N}$, which we shall fix. We shall also assume that $N \geq \epsilon^{-\frac{1}{2}}$ in order to guarantee $Q \geq 1$. In Section 6.3.1, we shall explicitly evaluate the sum $S_q(\boldsymbol{c})$ using Gauss sums. Next, in Section 6.3.2, we shall study the oscillatory integrals $I_q(\boldsymbol{c})$ using stationary phase. Finally, in Section 6.3.6, we shall combine the various estimates and complete the proof of Theorem 6.3.1.

6.3.1 *Gauss Sums and Kloosterman Sums*

In this section, we explicitly evaluate the exponential sum $S_q(\boldsymbol{c})$ in (6.30), for $\boldsymbol{c} \in \mathbb{Z}^4$ and relate it to the Kloosterman sum $S(m,n;c)$ in (3.11). The latter sum satisfies the well-known Weil bound

$$|S(m,n;c)| \leq \tau(c)\sqrt{(m,n,c)}\sqrt{c}, \tag{6.31}$$

where τ is the divisor function.

Recalling that $N \in 4\mathbb{N}$, it will be convenient to write $N = 4N'$ for $N' \in \mathbb{N}$. We have

$$S_q(\boldsymbol{c}) = \sum_{a \bmod q}^{*} e_q(-4aN') \prod_{i=1}^{4} \mathcal{G}(a, c_i; q), \tag{6.32}$$

where

$$\mathcal{G}(s,t;q) = \sum_{b \bmod q} e_q\left(sb^2 + tb\right),$$

for given non-zero integers s, t, q such that $q \geq 1$. The latter sum is classical and may be evaluated. Let

$$\delta_n = \begin{cases} 0, & \text{if } n \equiv 0 \bmod 2, \\ 1 & \text{if } n \equiv 1 \bmod 2, \end{cases} \qquad \epsilon_n = \begin{cases} 1, & \text{if } n \equiv 1 \bmod 4, \\ i, & \text{if } n \equiv 3 \bmod 4. \end{cases}$$

The following result is recorded in [BB12, Lemma 3], but goes back to Gauss.

Lemma 6.3.2. *Suppose that* $(s,q) = 1$. *Then,*

$$\mathcal{G}(s,t;q) = \begin{cases} \epsilon_q \sqrt{q} \left(\frac{s}{q}\right) e\left(-\frac{\overline{4s}t^2}{q}\right) & \textit{if } q \textit{ is odd}, \\ 2\delta_t \epsilon_v \sqrt{v} \left(\frac{2s}{v}\right) e\left(-\frac{\overline{8s}t^2}{v}\right) & \textit{if } q = 2v, \textit{ with } v \textit{ odd}, \\ (1+i)\epsilon_s^{-1}(1-\delta_t)\sqrt{q}\left(\frac{q}{s}\right) e\left(-\frac{\overline{s}t^2}{4q}\right) & \textit{if } 4 \mid q. \end{cases}$$

Our analysis of $S_q(\boldsymbol{c})$ now differs according to the 2-adic valuation of q. In each case, we shall be led to an appearance of the Kloosterman sum (3.10).

Suppose first that $q \equiv 1 \bmod 2$. By substituting Lemma 6.3.2 into (6.32), we directly obtain

$$S_q(\boldsymbol{c}) = q^2 \sum_{a \bmod q}^{*} e_q(-4aN' - \overline{4a}\|\boldsymbol{c}\|_2^2) = q^2 S(N', \|\boldsymbol{c}\|_2^2; q),$$

since $S(A, tB; q) = S(tA, B; q)$ for any $t \in (\mathbb{Z}/q\mathbb{Z})^*$.

If $q \equiv 2 \bmod 4$, then we write $q = 2v$, for odd v. This time, we obtain

$$\begin{aligned} S_q(\boldsymbol{c}) &= 2^4 \delta_{c_1c_2c_3c_4} v^2 \sum_{a \bmod q}^{*} e_q(-4aN') e_v(-\overline{8a}\|\boldsymbol{c}\|_2^2) \\ &= 4\delta_{c_1c_2c_3c_4} q^2 S(N', \|\boldsymbol{c}\|_2^2/4; v) \\ &= 4\delta_{c_1c_2c_3c_4} q^2 S(2N', \|\boldsymbol{c}\|_2^2/2; q), \end{aligned}$$

since $4 \mid \|\boldsymbol{c}\|_2^2$, when all the c_i are odd.

If $q \equiv 0 \bmod 4$, it follows from Lemma 6.3.2 that

$$S_q(\boldsymbol{c}) = -4(1-\delta_{c_1})\dots(1-\delta_{c_4}) q^2 \sum_{a \bmod q}^{*} e_q(-4aN') e_{4q}(-\overline{a}\|\boldsymbol{c}\|_2^2).$$

Thus, in this case, we find that

$$S_q(\boldsymbol{c}) = \begin{cases} 0 & \text{if } 2 \nmid \boldsymbol{c}, \\ -4q^2 S(N, \|\boldsymbol{c}'\|_2^2; q) & \text{if } \boldsymbol{c} = 2\boldsymbol{c}' \text{ for } \boldsymbol{c}' \in \mathbb{Z}^4. \end{cases}$$

6.3.2 *Oscillatory Integrals*

Recall the definition (6.30) of $I_q(\boldsymbol{c})$, in which w is given by (6.27). We make the change of variables $\boldsymbol{x} = \sqrt{N}\boldsymbol{x}'$ and $\boldsymbol{x}' = \boldsymbol{\xi} + \epsilon \boldsymbol{z}$. This leads to the expression

$$\begin{aligned} I_q(\boldsymbol{c}) &= N^2 \int_{\mathbb{R}^4} w(\boldsymbol{x}')\, h\left(\frac{q}{Q}, \frac{\|\boldsymbol{x}'\|_2^2 - 1}{\epsilon^2}\right) e_{\frac{q}{\sqrt{N}}}(-\boldsymbol{c}\cdot\boldsymbol{x}')\, d\boldsymbol{x}' \\ &= \epsilon^4 N^2 e_{\frac{q}{\sqrt{N}}}(-\boldsymbol{c}\cdot\boldsymbol{\xi}) \int_{\mathbb{R}^4} w_0(\|\boldsymbol{z}\|_2) w_0\left(\frac{2\boldsymbol{\xi}\cdot\boldsymbol{z}}{\epsilon}\right) h\left(\frac{q}{Q}, \frac{y(\boldsymbol{z})}{\epsilon}\right) e_{\frac{q}{\epsilon\sqrt{N}}}(-\boldsymbol{c}\cdot\boldsymbol{z})\, d\boldsymbol{z}, \end{aligned}$$

where $y(\boldsymbol{z}) = 2\boldsymbol{\xi}\cdot\boldsymbol{z} + \epsilon\|\boldsymbol{z}\|_2^2$. Let $r = q/Q$ and $\boldsymbol{v} = r^{-1}\boldsymbol{c}$. Then, we have

$$I_q(\boldsymbol{c}) = \epsilon^4 N^2 e_r(-\epsilon^{-1}\boldsymbol{c}\cdot\boldsymbol{\xi}) I_r^*(\boldsymbol{v}), \tag{6.33}$$

where

$$I_r^*(\boldsymbol{v}) = \int_{\mathbb{R}^4} w_0(\|\boldsymbol{x}\|_2) w_0\left(\frac{2\boldsymbol{\xi}\cdot\boldsymbol{x}}{\epsilon}\right) h\left(r, \frac{y(\boldsymbol{x})}{\epsilon}\right) e(-\boldsymbol{v}\cdot\boldsymbol{x})\, d\boldsymbol{x}. \tag{6.34}$$

In particular, we have $I_r^*(\boldsymbol{v}) = O(\epsilon/r)$, since $h(r,y) \ll r^{-1}$ and the region of integration has measure $O(\epsilon)$. Due to the change of variable, it will come in handy to define some new notation. For a vector $\boldsymbol{b} \in \mathbb{R}^4$, we define $\hat{b}_i$ by $\hat{b}_i = \boldsymbol{b}\cdot\boldsymbol{e}_i$, where $\boldsymbol{e}_i$ is a fixed orthonormal basis with $\boldsymbol{e}_4 = \boldsymbol{\xi}$. Thus, we have $\boldsymbol{b} = \sum_{i=1}^4 \hat{b}_i \boldsymbol{e}_i$.

6.3.3 *Easy Estimates*

Our attention now shifts to analysing $I_r^*(\boldsymbol{v})$ for $r \ll 1$ and $\boldsymbol{v} \in \mathbb{R}^4$. Let $\boldsymbol{x} \in \mathbb{R}^4$ such that $w_0(\|\boldsymbol{x}\|_2)w_0(2\boldsymbol{\xi}\cdot\boldsymbol{x}/\epsilon) \neq 0$. Then,

$$\frac{y(\boldsymbol{x})}{\epsilon} = \frac{2\boldsymbol{\xi}\cdot\boldsymbol{x} + \epsilon\|\boldsymbol{x}\|_2^2}{\epsilon} < 2.$$

Put $v(t) = w_0(t/6)$. Then, $v(y(\boldsymbol{x})/\epsilon) \gg 1$ whenever $w_0(\|\boldsymbol{x}\|_2)w_0(2\boldsymbol{\xi}\cdot\boldsymbol{x}/\epsilon) \neq 0$. We may now write

$$I_r^*(\boldsymbol{v}) = \frac{1}{r}\int_{\mathbb{R}^4} w_3(\boldsymbol{x}) f\left(\frac{y(\boldsymbol{x})}{\epsilon}\right) e(-\boldsymbol{v}\cdot\boldsymbol{x})\, d\boldsymbol{x},$$

where $f(y) = v(y)rh(r,y)$ and

$$w_3(\boldsymbol{x}) = \frac{w_0(\|\boldsymbol{x}\|_2)w_0(2\boldsymbol{\xi}\cdot\boldsymbol{x}/\epsilon)}{v(y(\boldsymbol{x})/\epsilon)}. \tag{6.35}$$

Let $p(t) = \mathcal{F}[f](t)$ be the Fourier transform of f. Then, the proof of [HB96, Lemma 17] shows that

$$p(t) \ll_j r(r|t|)^{-j}, \tag{6.36}$$

for any $j > 0$. We may therefore write

$$I_r^*(\boldsymbol{v}) = \frac{1}{r}\int_{\mathbb{R}} p(t) \int_{\mathbb{R}^4} w_3(\boldsymbol{x}) e\left(t\frac{y(\boldsymbol{x})}{\epsilon} - \boldsymbol{v}\cdot\boldsymbol{x}\right) d\boldsymbol{x}\, dt. \tag{6.37}$$

Building on this, we proceed by establishing the following result.

Lemma 6.3.3. *Let $\boldsymbol{c} \in \mathbb{Z}^4$, with $\boldsymbol{c} \neq \boldsymbol{0}$. Then,*

$$I_q(\boldsymbol{c}) \ll_j \frac{\epsilon^5 N^2 Q}{q} \min_{i=1,2,3}\left\{|\hat{c}_i|^{-j}, (\epsilon|\hat{c}_4|)^{-j}\right\},$$

for any $j > 0$.

This result corresponds to [Sar15a, Lemma 6.1]. Since $\max_i |\hat{c}_i| \gg \|\boldsymbol{c}\|_2$, it follows that

$$I_q(\boldsymbol{c}) \ll_j \frac{\epsilon^5 N^2 Q}{q}(\epsilon\|\boldsymbol{c}\|_2)^{-j},$$

for any $j > 0$. In this way, for any $\delta > 0$, Lemma 6.3.3 implies that there is a negligible contribution to (6.29) from $\boldsymbol{c}$ such that either of the inequalities $\|\boldsymbol{c}\|_2 > N^\delta/\epsilon$ or $\max_{i=1,2,3}\{|\hat{c}_i|, \epsilon|\hat{c}_4|\} > N^\delta$ hold. Thus, in (6.29), the summation over $\boldsymbol{c}$ can henceforth be restricted to the set $\mathcal{C}$, which is defined to be the set of $\boldsymbol{c} \in \mathbb{Z}^4$ for which $\max_{i=1,2,3}\{|\hat{c}_i|, \epsilon|\hat{c}_4|\} \leq N^\delta$. It follows from [Sar15a, Lemma 6.3] that $\#\mathcal{C} = O(\epsilon^{-1}N^{4\delta})$.

Proof of Lemma 6.3.3. We make the change of variables $\boldsymbol{x} = \sum_{i=1}^4 u_i \boldsymbol{e}_i$ in (6.37). Let $\boldsymbol{v} = \sum_{i=1}^4 \hat{v}_i \boldsymbol{e}_i$, where $\hat{v}_i = \boldsymbol{v} \cdot \boldsymbol{e}_i$. Then, on recalling (6.35), we find that

$$\begin{aligned} I_r^{\star}(\boldsymbol{v}) &= \frac{1}{r} \int_{\mathbb{R}} p(t) \int_{\mathbb{R}^4} w_3 \left(\sum_{i=1}^4 u_i \boldsymbol{e}_i \right) e \left(\frac{ty(\sum_{i=1}^4 u_i \boldsymbol{e}_i)}{\epsilon} - \boldsymbol{u} \cdot \hat{\boldsymbol{v}} \right) d\boldsymbol{u}\, dt \\ &= \frac{1}{r} \int_{\mathbb{R}} p(t) \int_{\mathbb{R}^4} \frac{w_0(\|\boldsymbol{u}\|_2) w_0(2u_4/\epsilon)}{v((2u_4 + \epsilon\|\boldsymbol{u}\|_2^2)/\epsilon)} e\left(F(\boldsymbol{u})\right) d\boldsymbol{u}\, dt, \end{aligned}$$

where $F(\boldsymbol{u}) = \frac{t}{\epsilon}\left\{2u_4 + \epsilon\|\boldsymbol{u}\|_2^2\right\} - \boldsymbol{u} \cdot \hat{\boldsymbol{v}}$. We have

$$\frac{\partial F(\boldsymbol{u})}{\partial u_i} = \begin{cases} 2tu_i - \hat{v}_i & \text{if } 1 \le i \le 3, \\ 2tu_4 - \hat{v}_4 + \frac{2t}{\epsilon} & \text{if } i = 4. \end{cases}$$

The proof of the lemma now follows from repeated integration by parts in conjunction with (6.36), much as in the proof of [HB96, Lemma 19]. Thus, when $i \in \{1,2,3\}$, integration by parts with respect to u_i readily yields

$$I_r^*(\boldsymbol{v}) \ll_j \frac{\epsilon}{r} \left\{ r|\hat{v}_i|^{1-j} + r^{1-j}|\hat{v}_i|^{1-j} \right\} \ll_j \epsilon r^{-j} |\hat{v}_i|^{1-j},$$

for any $j > 0$, since $r \ll 1$. Likewise, integrating by parts with respect to u_4, we get

$$I_r^*(\boldsymbol{v}) \ll_j \frac{\epsilon}{r} \left\{ r(\epsilon|\hat{v}_4|)^{1-j} + r^{1-j}(\epsilon|\hat{v}_4|)^{1-j} \right\} \ll_j \epsilon r^{-j} (\epsilon|\hat{v}_4|)^{1-j}.$$

The statement of the lemma follows on recalling (6.33) and the fact that $\boldsymbol{c} = r\boldsymbol{v}$, with $r = q/Q$. ∎

6.3.4 *Stationary Phase*

The following stationary phase result will prove vital in our more demanding analysis of $I_q(\boldsymbol{c})$ in the next section.

Lemma 6.3.4. *Let ϕ be a Schwartz function on $\mathbb{R}^n$ and let $N \ge 0$. Then,*

$$\int_{\mathbb{R}^n} e^{i\lambda\|\boldsymbol{x}\|_2^2} \phi(\boldsymbol{x}) d\boldsymbol{x} = \lambda^{-\frac{n}{2}} \sum_{j=0}^{N} a_j \lambda^{-j} + O_{n,N}\left(|\lambda|^{-\frac{n}{2}-N-1} \|\phi\|_{2N+3+n,1} \right),$$

where $\|\cdot\|_{k,1}$ denotes the Sobolev norm on $L^1(\mathbb{R}^n)$ of order k and

$$a_j = (i\pi)^{\frac{n}{2}} \frac{i^j}{j!} \left(\Delta_{\mathbb{R}^n}^j \phi \right)(\boldsymbol{0}).$$

Proof. We follow the argument in Stein [Ste93, §VIII.5.1]. By using the Fourier transform, we can write the integral as

$$\left(\frac{i\pi}{\lambda}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{-i\pi^2\|\boldsymbol{\xi}\|_2^2/\lambda} \mathcal{F}[\phi](\boldsymbol{\xi}) d\boldsymbol{\xi}. \tag{6.38}$$

Next, we split off the first N terms in a Taylor expansion around $\mathbf{0}$, finding that

$$e^{-i\pi^2\|\boldsymbol{\xi}\|_2^2/\lambda} = \sum_{j=0}^{N} \frac{(-i\pi^2\|\boldsymbol{\xi}\|_2^2/\lambda)^j}{j!} + R_N(\boldsymbol{\xi}).$$

The main term now comes from integration by parts and Fourier inversion. We are left to deal with the integral involving $R_N(\boldsymbol{\xi})$. We have

$$R_N(\boldsymbol{\xi}) \ll_N \left(\frac{\|\boldsymbol{\xi}\|_2^2}{|\lambda|}\right)^{N+1}, \tag{6.39}$$

which follows from Taylor expansion when $\|\boldsymbol{\xi}\|_2^2 \leq |\lambda|$ and trivially otherwise. Moreover,

$$\mathcal{F}[\phi](\boldsymbol{\xi}) = O_A\left(\|\boldsymbol{\xi}\|_2^{-A}\|\phi\|_{A,1}\right), \tag{6.40}$$

for any $A \geq 0$. We split up the remaining integral into two parts: $\|\boldsymbol{\xi}\|_2 \leq 1$ and $\|\boldsymbol{\xi}\|_2 > 1$. For the first part, we use (6.39) and (6.40) with $A = 2N+1+n$. Recalling the additional factor $\lambda^{-\frac{n}{2}}$ from (6.38), we get an error term of size

$$O_{n,N}\left(|\lambda|^{-\frac{n}{2}-N-1}\|\phi\|_{2N+1+n,1}\right).$$

For the second part, we use (6.39) and (6.40), but this time with $A = 2N+3+n$. This leads to the same overall error term, but with the factor $\|\phi\|_{2N+1+n,1}$ replaced by $\|\phi\|_{2N+3+n,1}$. □

6.3.5 *Hard Estimates*

Having shown how to truncate the sum over $\boldsymbol{c}$ in (6.29), we now return to (6.33) for $\boldsymbol{c} \in \mathcal{C}$ and see what more can be said about the integral $I_r^*(\boldsymbol{v})$ in (6.34), with $r = q/Q$ and $\boldsymbol{v} = r^{-1}\boldsymbol{c}$. Our result relies on an asymptotic expansion of $I_r^*(\boldsymbol{v})$, but the form it takes depends on the size of $\epsilon|\hat{v}_4|$.

It will be convenient to set $\boldsymbol{a} = (\hat{v}_1, \hat{v}_2, \hat{v}_3)$, in what follows. To begin with, we make the change of variables $\boldsymbol{x} = \sum_{i=1}^4 u_i \boldsymbol{e}_i$ in (6.34). This leads to the expression

$$I_r^*(\boldsymbol{v}) = \int_{\mathbb{R}^4} w_0(\|\boldsymbol{u}\|_2) w_0(2u_4/\epsilon) h\left(r, \frac{2u_4}{\epsilon} + \|\boldsymbol{u}\|_2^2\right) e(-\boldsymbol{u}\cdot\hat{\boldsymbol{v}})\, d\boldsymbol{u},$$

where $\hat{v}_i = \boldsymbol{v} \cdot \boldsymbol{e}_i$ for $1 \leq i \leq 4$. We now write $y = 2u_4/\epsilon + \|\boldsymbol{u}\|_2^2$, under which we have

$$u_4 = \frac{1}{\epsilon}\left(-1 + \sqrt{1 + \epsilon^2\{y - u_1^2 - u_2^2 - u_3^2\}}\right). \tag{6.41}$$

Thus,

$$I_r^*(\boldsymbol{v}) = \int_{\mathbb{R}} h(r,y) e\left(-\frac{\epsilon \hat{v}_4 y}{2}\right) T(y) dy, \tag{6.42}$$

where

$$T(y) = e\left(\frac{\epsilon \hat{v}_4 y}{2}\right) \int_{\mathbb{R}^3} w_0(\|\boldsymbol{u}\|_2) w_0(2u_4/\epsilon) e(-\boldsymbol{u} \cdot \hat{\boldsymbol{v}}) \frac{du_1 du_2 du_3}{2/\epsilon + 2u_4}, \tag{6.43}$$

and u_4 is given in terms of y, u_1, u_2, u_3 by (6.41). In particular, on writing $\boldsymbol{x} = (u_1, u_2, u_3)$, we have $w_0(\|\boldsymbol{u}\|_2) w_0(2u_4/\epsilon) = \psi_y(\boldsymbol{x})$, where $\psi_y : \mathbb{R}^3 \to \mathbb{R}_{\geq 0}$ is the weight function

$$\begin{aligned}\psi_y(\boldsymbol{x}) = w_0 &\left(2\epsilon^{-2}(-1 + \sqrt{1 + \epsilon^2\{y - \|\boldsymbol{x}\|_2^2\}})\right) \\ &\times w_0\left(\sqrt{\|\boldsymbol{x}\|_2^2 + \epsilon^{-2}(1 - \sqrt{1 + \epsilon^2\{y - \|\boldsymbol{x}\|_2^2\}})^2}\right).\end{aligned} \tag{6.44}$$

We note, furthermore, that the integral in $T(y)$ is supported on $[-1,1]^3$. Moreover, we have

$$\frac{2u_4}{\epsilon} = \frac{2}{\epsilon^2}\left(-1 + \sqrt{1 + \epsilon^2\{y - \|\boldsymbol{x}\|_2^2\}}\right) = y - \|\boldsymbol{x}\|_2^2 + O(\epsilon^2), \tag{6.45}$$

for any $\boldsymbol{x}$ such that $\psi_y(\boldsymbol{x}) \neq 0$. In particular, it follows that

$$\frac{1}{2/\epsilon + 2u_4} = \frac{\epsilon}{2}\left(1 + O(\epsilon^2)\right) \tag{6.46}$$

in (6.43).

Since $e(z) = 1 + O(z)$, we invoke (6.41) and (6.45) to deduce that

$$e(-\boldsymbol{u} \cdot \hat{\boldsymbol{v}}) = e\left(-\frac{\epsilon \hat{v}_4 y}{2}\right) e\left(\frac{\epsilon \hat{v}_4}{2}\|\boldsymbol{x}\|_2^2 - \boldsymbol{a} \cdot \boldsymbol{x}\right)\left(1 + O(|\epsilon \hat{v}_4|\epsilon^2)\right), \tag{6.47}$$

where we recall that $\boldsymbol{a} = (\hat{v}_1, \hat{v}_2, \hat{v}_3)$. Thus, it follows from (6.46) that

$$T(y) = \frac{\epsilon}{2}\left(1 + O(\epsilon^2 + |\epsilon \hat{v}_4|\epsilon^2)\right) I(y), \tag{6.48}$$

where

$$I(y) = \int_{\mathbb{R}^3} \psi_y(\boldsymbol{x}) e\left(\frac{\epsilon \hat{v}_4}{2}\|\boldsymbol{x}\|_2^2 - \boldsymbol{a} \cdot \boldsymbol{x}\right) d\boldsymbol{x}. \tag{6.49}$$

In what follows, it will be useful to record the estimate

$$\int_{\mathbb{R}} \left| r^k y^\ell \frac{\partial^k h(r,y)}{\partial r^k} \right| dy \ll_\ell r^\ell, \tag{6.50}$$

for any $\ell \geq 0$ and $k \in \{0,1\}$. This is a straightforward consequence of Lemma 5.2.3. The stage is now set to prove the following preliminary estimate for $I_r^*(\boldsymbol{v})$ and its partial derivative with respect to r.

Lemma 6.3.5. *Let $k \in \{0,1\}$. Then,*

$$r^{2k}\frac{\partial^k I_r^*(\boldsymbol{v})}{\partial r^k} \ll \frac{\epsilon(1+\epsilon^3|\hat{v}_4|)}{\max\{1,(\epsilon|\hat{v}_4|)\}^{\frac{3}{2}}}N^{\delta}.$$

Proof. Suppose first that $k=0$. An application of [HBP17, Lemmata 3.1 and 3.2] shows that

$$I(y) \ll \frac{1}{\max\{1,(\epsilon|\hat{v}_4|)\}^{\frac{3}{2}}},$$

since $\|\mathcal{F}[\psi_y]\|_1 \ll 1$. The desired bound now follows on substituting this into (6.42) and (6.48), before using (6.50) with $k=\ell=0$ to carry out the integration over y.

Suppose next that $k=1$. Then, in view of (6.42), we have

$$\begin{aligned} r^2\frac{\partial I_r^*(\boldsymbol{v})}{\partial r} &= \int_{\mathbb{R}} r^2\frac{\partial h(r,y)}{\partial r} e\left(-\frac{\epsilon\hat{v}_4 y}{2}\right)T(y)dy \\ &\quad + \int_{\mathbb{R}} h(r,y)e\left(-\frac{\epsilon\hat{v}_4 y}{2}\right)\widetilde{T}(y)dy, \end{aligned} \tag{6.51}$$

where

$$\begin{aligned} \widetilde{T}(y) &= e\left(\frac{\epsilon\hat{v}_4 y}{2}\right)\int_{\mathbb{R}^3} w_0(\|\boldsymbol{u}\|_2)w_0(2u_4/\epsilon)r^2\frac{\partial}{\partial r}e(-\boldsymbol{u}\cdot\hat{\boldsymbol{v}})\frac{du_1du_2du_3}{2/\epsilon+2u_4} \\ &= e\left(\frac{\epsilon\hat{v}_4 y}{2}\right)\int_{\mathbb{R}^3}(2\pi i\boldsymbol{u}\cdot\hat{\boldsymbol{c}})w_0(\|\boldsymbol{u}\|_2)w_0(2u_4/\epsilon)e(-\boldsymbol{u}\cdot\hat{\boldsymbol{v}})\frac{du_1du_2du_3}{2/\epsilon+2u_4}. \end{aligned}$$

The contribution from the first integral in (6.51) is satisfactory, since $r \ll 1$, on reapplying our argument for $k=0$ and using (6.50) with $k=1$ and $\ell=0$. Turning to the second integral in (6.51), we recall (6.46) and (6.47). These allow us to write

$$\widetilde{T}(y) = \epsilon\pi i\left(1+O(\epsilon^2+|\epsilon\hat{v}_4|\epsilon^2)\right)\widetilde{I}(y),$$

where

$$\widetilde{I}(y) = \int_{\mathbb{R}^3}\widetilde{\psi_y}(\boldsymbol{x})e\left(\frac{\epsilon\hat{v}_4}{2}\|\boldsymbol{x}\|_2^2-\boldsymbol{a}\cdot\boldsymbol{x}\right)d\boldsymbol{x}$$

and

$$\widetilde{\psi_y}(\boldsymbol{x}) = \left(r\boldsymbol{a}\cdot\boldsymbol{x}+\frac{\hat{c}_4}{\epsilon}\left(-1+\sqrt{1+\epsilon^2\{y-\|\boldsymbol{x}\|_2^2\}}\right)\right)\psi_y(\boldsymbol{x}).$$

Here, the definition of $\mathcal{C}$ implies that $r|\boldsymbol{a}| = \max\{|\hat{c}_1|,|\hat{c}_2|,|\hat{c}_3|\} \le N^{\delta}$ and $\epsilon|\hat{c}_4| \le N^{\delta}$. Thus, the L^1-norm of the Fourier transform of $\widetilde{\psi_y}$ is $O(N^{\delta})$. Once combined with (6.50) with $k=\ell=0$, we apply [HBP17, Lemmata 3.1 and 3.2] to estimate $\widetilde{I}(y)$, which concludes our treatment of the case $k=1$. □

The case $k=0$ of Lemma 6.3.5 is already implicit in Sardari's work (see [Sar15a, Lemma 6.2]). We shall also need the case $k=1$, but it turns out that it is only effective when r is essentially of size 1. For general r, we require a pair of asymptotic expansions for $I_r^*(\boldsymbol{v})$, that are relevant for small and large values of $\epsilon|\hat{v}_4|$, respectively. This is the objective of the following pair of results.

Lemma 6.3.6. *Let $A \geq 0$. Then,*

$$I_r^*(\boldsymbol{v}) = \frac{\epsilon I(0)}{2} + O_A\left(\epsilon^3(1+\epsilon|\hat{v}_4|) + \epsilon(1+\epsilon|\hat{v}_4|)^A r^A\right).$$

Proof. Our first approach is founded on the Taylor expansion

$$e\left(-\frac{\epsilon\hat{v}_4 y}{2}\right) = \sum_{j=0}^{A-1} \frac{(-\pi i\epsilon\hat{v}_4 y)^j}{j!} + R_A(y),$$

where $R_A(y) \ll_A (\epsilon|\hat{v}_4 y|)^A$. Since $I(y) \ll 1$, we conclude from (6.42), (6.48) and (6.50) that

$$I_r^*(\boldsymbol{v}) = \frac{\epsilon}{2}\sum_{j=0}^{A-1} \frac{(-\pi i\epsilon\hat{v}_4)^j}{j!} \int_{\mathbb{R}} y^j h(r,y) I(y)dy$$
$$+ O_A\left(\epsilon^3(1+\epsilon|\hat{v}_4|) + \epsilon(\epsilon|\hat{v}_4|)^A r^A\right).$$

Next, we claim that

$$\int_{\mathbb{R}} y^j h(r,y)I(y)dy = O_A(r^A) + \begin{cases} I(0) & \text{if } j=0, \\ 0 & \text{if } j>0. \end{cases} \tag{6.52}$$

To see this, note that $y^j I(y)$ has uniformly bounded Sobolev norms (in terms of $\boldsymbol{c}$ and ϵ) and apply Lemma 5.2.2. The statement of the lemma is now obvious. □

Lemma 6.3.7. *Assume that $\epsilon|\hat{v}_4| > 1$. For each $j \geq 0$, we define*

$$\varphi_j(y) = \Delta^j_{\mathbb{R}^3}\psi_y\left((\epsilon\hat{v}_4)^{-1}\boldsymbol{a}\right) = \Delta^j_{\mathbb{R}^3}\psi_y\left((\epsilon\hat{c}_4)^{-1}(\hat{c}_1,\hat{c}_2,\hat{c}_3)\right),$$

where ψ_y is given by (6.44). Let $A \geq 0$. Then, there exist constants k_j that depend only on j such that

$$I_r^*(\boldsymbol{v}) = \frac{\epsilon\delta(\hat{\boldsymbol{c}})}{(\epsilon\hat{v}_4)^{\frac{3}{2}}} e\left(-\frac{\|\boldsymbol{a}\|_2^2}{2\epsilon\hat{v}_4}\right)\sum_{j=0}^{A}\frac{k_j}{(\epsilon\hat{v}_4)^j}\int_{\mathbb{R}} h(r,y)e\left(-\frac{\epsilon\hat{v}_4 y}{2}\right)\varphi_j(y)dy$$
$$+ O_A\left(\frac{\epsilon^3}{|\epsilon\hat{v}_4|^{\frac{1}{2}}} + \frac{\epsilon}{|\epsilon\hat{v}_4|^{\frac{5}{2}+A}}\right),$$

where

$$\delta(\hat{\boldsymbol{c}}) = \begin{cases} 1 & \text{if } \epsilon|\hat{c}_4| \gg \|(\hat{c}_1,\hat{c}_2,\hat{c}_3)\|_2, \\ 0 & \text{otherwise}. \end{cases}$$

Proof. It will be convenient to set $\lambda = \epsilon\hat{v}_4$ in the proof of this result, recalling our hypothesis that $|\lambda| > 1$. Our starting point is the expression for $T(y)$ in (6.48), in which $I(y)$ is given by (6.49). By completing the square, we may write

$$T(y) = \frac{\epsilon}{2}\left(1 + O(|\lambda|\epsilon^2)\right) e\left(-\frac{\|\boldsymbol{a}\|_2^2}{2\lambda}\right) I^*(y),$$

since $|\lambda| > 1$, where

$$I^*(y) = \int_{\mathbb{R}^3} \psi_y\left(\boldsymbol{x} + \frac{\boldsymbol{a}}{\lambda}\right) e\left(\frac{\lambda}{2}\|\boldsymbol{x}\|_2^2\right) d\boldsymbol{x}.$$

If $\|\boldsymbol{a}\|_2 \gg \epsilon|\hat{v}_4|$, then it follows from [HB96, Lemma 10] that $T(y) \ll_A \epsilon|\lambda|^{-A}$, for any $A \geq 0$. Alternatively, if $\|\boldsymbol{a}\|_2 \ll \epsilon|\hat{v}_4|$, which is equivalent to $\delta(\hat{\boldsymbol{c}}) = 1$, then all the hypotheses of Lemma 6.3.4 are met. Thus, for any $A \geq 0$, there exist constants k_j that depend only on j such that

$$I^*(y) = \frac{1}{\lambda^{\frac{3}{2}}} \sum_{j=0}^{A} \frac{k_j \Delta^j_{\mathbb{R}^3} \psi_y(\lambda^{-1}\boldsymbol{a})}{\lambda^j} + O_A\left(\frac{1}{|\lambda|^{\frac{5}{2}+A}}\right).$$

Hence, we conclude from (6.48) that

$$T(y) = \frac{\epsilon\delta(\hat{\boldsymbol{c}})}{2\lambda^{\frac{3}{2}}} e\left(-\frac{\|\boldsymbol{a}\|_2^2}{2\lambda}\right) \sum_{j=0}^{A} \frac{k_j \Delta^j_{\mathbb{R}^3} \psi_y(\lambda^{-1}\boldsymbol{a})}{\lambda^j} + O_A\left(\frac{\epsilon^3}{|\lambda|^{\frac{1}{2}}} + \frac{\epsilon}{|\lambda|^{\frac{5}{2}+A}}\right).$$

We now wish to substitute this into our expression (6.42) for $I_r^*(\boldsymbol{v})$. In order to control the contribution from the error term, we apply (6.50) with $\ell = 0$. We therefore arrive at the statement of the lemma on redefining k_j to be $k_j/2$. □

It remains to consider the integral

$$\begin{aligned} J_{j,q}(\boldsymbol{c}) &= \int_{\mathbb{R}} h\left(r, y\right) e\left(-\frac{\epsilon\hat{v}_4 y}{2}\right) \varphi_j(y) dy \\ &= \int_{\mathbb{R}} h\left(\frac{q}{Q}, y\right) e\left(-\frac{\epsilon\hat{c}_4 y Q}{2q}\right) \varphi_j(y) dy, \end{aligned} \tag{6.53}$$

for $j \geq 0$. Recollecting (6.44), all we shall need to know about φ_j is that it is a smooth compactly supported function with bounded derivatives, and that it does not depend on q. (Note that we may assume that $|(\hat{c}_1, \hat{c}_2, \hat{c}_3)| \ll \epsilon|\hat{c}_4|$ in what follows, since otherwise $\delta(\hat{\boldsymbol{c}}) = 0$.)

Lemma 6.3.8. *Let $\boldsymbol{c} \in \mathcal{C}$ and $k \in \{0, 1\}$. Then,*

$$q^k \frac{\partial^k J_{j,q,}(\boldsymbol{c})}{\partial q^k} \ll_j N^\delta.$$

Proof. When $k=0$, the result follows immediately from (6.50). Suppose next that $k=1$. Then, (6.53) implies that

$$\begin{aligned}\frac{\partial J_{j,q}(\boldsymbol{c})}{\partial q} &= \frac{1}{Q}\int_{\mathbb{R}} \frac{\partial h\left(r,y\right)}{\partial r} e\left(-\frac{\epsilon \hat{c}_4 yQ}{2q}\right)\varphi_j(y)dy \\ &\quad + \int_{\mathbb{R}} \frac{\pi i \epsilon \hat{c}_4 yQ}{q^2} h\left(r,y\right) e\left(-\frac{\epsilon \hat{c}_4 yQ}{2q}\right)\varphi_j(y)dy \\ &= J_1 + J_2,\end{aligned}$$

say. It follows from (6.50) that $J_1 \ll_j Q^{-1}r^{-1} = q^{-1}$, which is satisfactory. Next, a further application of (6.50) yields

$$J_2 \ll_j \frac{\epsilon|\hat{c}_4|Q}{q^2}\int_{\mathbb{R}} |yh\left(r,y\right)|\,dy \ll_j \frac{\epsilon|\hat{c}_4|Q}{q^2}\cdot r \leq \frac{N^\delta}{q},$$

for $\boldsymbol{c}\in\mathcal{C}$. □

6.3.6 *Putting everything together*

It is now time to return to (6.29), in order to conclude the proof of Theorem 6.3.1.

6.3.7 *The Main Term*

We begin by dealing with the main contribution, which comes from the term $\boldsymbol{c}=\mathbf{0}$. Denoting this by $M(w)$, we see that

$$M(w) = \frac{1}{Q^2}\sum_{q\ll Q} q^{-4}S_q(\mathbf{0})I_q(\mathbf{0}) + O_A(Q^{-A}), \tag{6.54}$$

for any $A>0$.

In view of (6.44), $\psi_0(\boldsymbol{x})$ is equal to

$$w_0\left(2\epsilon^{-2}(-1+\sqrt{1-\epsilon^2\|\boldsymbol{x}\|_2^2})\right)w_0\left(\sqrt{\|\boldsymbol{x}\|_2^2+\epsilon^{-2}(1-\sqrt{1-\epsilon^2\|\boldsymbol{x}\|_2^2})^2}\right).$$

As in (6.45), when $\psi_0(\boldsymbol{x})\neq 0$ we must have

$$2\epsilon^{-2}\left(-1+\sqrt{1-\epsilon^2\|\boldsymbol{x}\|_2^2}\right) = -\|\boldsymbol{x}\|_2^2 + O(\epsilon^2),$$
$$\|\boldsymbol{x}\|_2^2+\epsilon^{-2}(1-\sqrt{1-\epsilon^2\|\boldsymbol{x}\|_2^2})^2 = \|\boldsymbol{x}\|_2^2 + O(\epsilon^2).$$

In particular, it is clear that

$$\sigma_\infty = \int_{\mathbb{R}^3}\psi_0(\boldsymbol{x})d\boldsymbol{x} \gg 1, \tag{6.55}$$

for an absolute implied constant. We now establish the following result.

Lemma 6.3.9. *We have*

$$I_q(\mathbf{0}) = \tfrac{1}{2}\epsilon^5 N^2 \sigma_\infty \left(1 + O(\epsilon^2) + O_A\left((q/Q)^A\right)\right),$$

for any $A > 0$, where σ_∞ is given by (6.55).

Proof. Returning to (6.33), it follows from (6.42) and (6.43) that

$$I_q(\mathbf{0}) = \epsilon^4 N^2 \int_{\mathbb{R}} h(r,y) K(y) dy,$$

where

$$K(y) = \int_{\mathbb{R}^3} w_0(\|\boldsymbol{u}\|_2) w_0(2u_4/\epsilon) \frac{du_1 du_2 du_3}{2/\epsilon + 2u_4},$$

and u_4 is given in terms of y, u_1, u_2, u_3 by (6.41). By using (6.46), we may write

$$K(y) = \frac{\epsilon}{2}\left(1 + O(\epsilon^2)\right) K^*(y), \quad \text{with } K^*(y) = \int_{\mathbb{R}^3} \psi_y(\boldsymbol{x}) d\boldsymbol{x}.$$

From (6.55), we see that $K^*(0) = \sigma_\infty$. This together with Lemma 5.2.2 yields

$$\int_{\mathbb{R}} h(r,y) K^*(y) dy = \sigma_\infty + O_A(r^A),$$

for any $A > 0$, since the Sobolev norms are uniformly bounded again (in terms of ϵ). We therefore deduce that

$$I_q(\mathbf{0}) = \tfrac{1}{2}\epsilon^5 N^2 \sigma_\infty \left(1 + O(\epsilon^2) + O_A(r^A)\right),$$

which completes the proof of the lemma. □

Now, it is clear from Section 6.3.1 that $q^{-4}|S_q(\boldsymbol{c})| \leq 4q^{-2}|S(m,n;q)|$, for any vector $\boldsymbol{c} \in \mathbb{Z}^4$, where (m,n) is $(N, \|\hat{\mathbf{c}}\|_2^2/4)$, $(N/2, \|\hat{\mathbf{c}}\|_2^2/2)$ or $(N/4, \|\hat{\mathbf{c}}\|_2^2)$ depending on whether $4 \mid q$, $q \equiv 2 \bmod 4$ or $2 \nmid q$, respectively. Hence, it follows from (6.31), together with the standard estimate for the divisor function, that

$$\sum_{t/2<q\leq t} q^{-4}|S_q(\boldsymbol{c})| \ll \sum_{t/2<q\leq t} q^{-2}|S(m,n;q)| \ll t^{o(1)} \sum_{t/2<q\leq t} \frac{\sqrt{(q,N)}}{q^{3/2}} \ll t^{-1/2+o(1)} N^{o(1)}, \tag{6.56}$$

for any $t > 1$. Returning to (6.54), we may now conclude from Lemma 6.3.9 and (6.56) with $\boldsymbol{c} = \mathbf{0}$, that the contribution to $M(w)$ from $q \leq Q^{1-\delta}$ is

$$\begin{aligned} &= \frac{1}{Q^2} \sum_{q \leq Q^{1-\delta}} q^{-4} S_q(\mathbf{0}) I_q(\mathbf{0}) + O_A(Q^{-A}) \\ &= \frac{\epsilon^5 N^2}{2Q^2} \sigma_\infty \mathfrak{S}(Q^{1-\delta}) + O\left(\frac{\epsilon^7 N^{2+\delta/2}}{Q^2}\right) + O_A(Q^{-A}), \end{aligned}$$

where

$$\mathfrak{S}(t) = \sum_{q \le t} q^{-4} S_q(\mathbf{0}).$$

This sum is absolutely convergent and satisfies $\mathfrak{S}(t) = \mathfrak{S} + O(t^{-1/2+o(1)} N^{o(1)})$, by (6.56). Here, in the usual way, $\mathfrak{S}$ is the Hardy–Littlewood product of local densities recorded in (6.28).

Next, on invoking (6.56), once more, the contribution from $q > Q^{1-\delta}$ is

$$\ll_A \frac{\epsilon^5 N^2}{Q^2} \sum_{q > Q^{1-\delta}} q^{-4} |S_q(\mathbf{0})| + Q^{-A} \ll \frac{\epsilon^5 N^{2+\delta/2} Q^{\delta/2}}{Q^{5/2}}.$$

Hence, we have established the following result, on recalling that $Q = \epsilon\sqrt{N}$, which shows that the main term is satisfactory for Theorem 6.3.1.

Lemma 6.3.10. *We have*

$$M(w) = \frac{\epsilon^3 N \sigma_\infty \mathfrak{S}}{2} + O\left(\epsilon^5 N^{1+o(1)} + \epsilon^{\frac{5}{2}} N^{\frac{3}{4}+o(1)}\right).$$

6.3.8 *The Error Term*

It remains to analyse the contribution $E(w)$, say, to $\Sigma(w)$ from vectors $\boldsymbol{c} \neq \mathbf{0}$ in (6.29). According to our work in Section 6.3.1 the value of $S_q(\boldsymbol{c})$ differs according to the residue class of q modulo 4. We have

$$E(w) = \sum_{i \bmod 4} E_i(w),$$

where $E_i(w)$ denotes the contribution from $q \equiv i \bmod 4$. Recall the definition of $\mathcal{C}$ from after the statement of Lemma 6.3.3. In order to unify our treatment of the four cases, we write $\mathcal{C}_1 = \mathcal{C}_2 = \mathcal{C}$ and we denote by $\mathcal{C}_2$ (resp. $\mathcal{C}_4$) the set of $\boldsymbol{c} \in \mathcal{C}$ for which $2 \nmid c_1 \dots c_4$ (resp. $2 \mid \boldsymbol{c}$). It will also be convenient to set

$$(m_1, n_1) = (m_3, n_3) = (N/4, \|\boldsymbol{c}\|_2^2),$$
$$(m_2, n_2) = (N/2, \|\boldsymbol{c}\|_2^2/2), \quad (m_4, n_4) = (N, \|\boldsymbol{c}\|_2^2/4).$$

In particular, $m_i n_i = N\|\hat{\boldsymbol{c}}\|_2^2/4 > 0$ for $1 \le i \le 4$, since $\|\boldsymbol{c}\|_2 = \|\hat{\boldsymbol{c}}\|_2$.

Let $1 \ll R \ll Q$. We denote by $E_i(w,R)$ the overall contribution to $E_i(w)$ from $q \sim R$. (We write $q \sim R$ to denote $q \in (R/2, R]$.) On recalling (6.33), it follows from our work so far that

$$\begin{aligned} E_i(w,R) &\ll \frac{1}{Q^2} \sum_{\substack{\boldsymbol{c} \in \mathcal{C}_i \\ \boldsymbol{c} \neq \boldsymbol{0}}} \left| \sum_{\substack{q \sim R \\ q \equiv i \bmod 4}} q^{-2} S(m_i, n_i; q) I_q(\boldsymbol{c}) \right| \\ &\ll \frac{\epsilon^4 N^2}{Q^2} \sum_{\substack{\boldsymbol{c} \in \mathcal{C}_i \\ \boldsymbol{c} \neq \boldsymbol{0}}} \left| \sum_{\substack{q \sim R \\ q \equiv i \bmod 4}} q^{-2} S(m_i, n_i; q) e_r(-\epsilon^{-1} \boldsymbol{c} \cdot \boldsymbol{\xi}) I_r^*(\boldsymbol{v}) \right|. \end{aligned} \tag{6.57}$$

Contribution from large q

Suppose first that $R \geq Q^{1-\eta}$, for some small $\eta > 0$. (The choice $\eta = 2\delta$ is satisfactory.) We have

$$e_r(-\epsilon^{-1} \boldsymbol{c} \cdot \boldsymbol{\xi}) = e\left(\frac{2\sqrt{m_i n_i}}{q} \alpha \right),$$

with

$$|\alpha| = \epsilon^{-1} |\hat{c}_4| \cdot \frac{Q}{q} \cdot \frac{q}{2\sqrt{m_i n_i}} = \frac{|\hat{c}_4|}{\|\hat{\boldsymbol{c}}\|_2} \leq 1.$$

It now follows from Conjecture 4.0.2 that

$$L(t) = \sum_{\substack{q \leq t \\ q \equiv i \bmod 4}} \frac{S(m_i, n_i; q)}{q} e\left(\frac{2\sqrt{m_i n_i}}{q} \alpha \right) \ll (tN)^{o(1)}. \tag{6.58}$$

Applying partial summation, based on Lemma 6.3.5, we deduce that

$$\begin{aligned} E_i(w,R) &\ll \frac{\epsilon^5 N^{2+\delta+o(1)}}{Q^3} \cdot \frac{Q^2}{R^2} \cdot \sum_{\substack{\boldsymbol{c} \in \mathcal{C}_i \\ \boldsymbol{c} \neq \boldsymbol{0}}} \frac{1}{\max\{1, \epsilon |\hat{c}_4| Q/R\}^{\frac{3}{2}}} \\ &\ll \frac{\epsilon^5 N^{2+\delta+o(1)}}{QR^2} \cdot \frac{\epsilon^{-1} R}{Q} \\ &= \frac{\epsilon^4 N^{2+\delta+o(1)}}{Q^2 R}. \end{aligned}$$

Since $R \geq Q^{1-\eta}$, we deduce that

$$E_i(w,R) \ll \frac{\epsilon^4 N^{2+\delta+o(1)} Q^\eta}{Q^3} \leq \epsilon N^{\frac{1}{2}+\delta+o(1)+\eta}.$$

This is satisfactory for Theorem 6.3.1, provided that η and δ are small enough.

Contribution from small q and small $\epsilon|\hat{v}_4|$

For the rest of the proof we suppose that $R < Q^{1-\eta}$. Let us put

$$\boldsymbol{b} = (\hat{c}_1, \hat{c}_2, \hat{c}_3),$$

so that $\boldsymbol{a} = r^{-1}\boldsymbol{b}$ in Lemmata 6.3.6 and 6.3.7. Let $E_i^{(\text{small})}(w, R)$ denote the contribution to $E_i(w, R)$ from $\boldsymbol{c} \in \mathcal{C}_i$ such that

$$\epsilon|\hat{c}_4| \leq \frac{R^{1+\delta}}{Q}. \tag{6.59}$$

In this case, it is advantageous to apply Lemma 6.3.6 to evaluate $I_r^*(\boldsymbol{v})$. To begin with, we consider the effect of substituting the main term from Lemma 6.3.6. Noting that $(\epsilon\hat{v}_4)^{-1}\boldsymbol{a} = (\epsilon\hat{c}_4)^{-1}\boldsymbol{b}$ does not depend on q, we deduce from (6.49) that the only dependence on q in $I(y)$ comes through the term

$$e\left(\frac{\epsilon\hat{v}_4}{2}\|\boldsymbol{x}\|_2^2 - \boldsymbol{a}\cdot\boldsymbol{x}\right) = e_r\left(\frac{\epsilon\hat{c}_4}{2}\|\boldsymbol{x}\|_2^2 - \boldsymbol{b}\cdot\boldsymbol{x}\right),$$

in the integrand. Thus, the main term in Lemma 6.3.6 makes the overall contribution

$$\ll \frac{\epsilon^5 N^2}{Q^2} \sum_{\substack{\boldsymbol{c}\in\mathcal{C}_i \\ \boldsymbol{c}\neq\boldsymbol{0} \\ (6.59)\text{ holds}}} \left| \sum_{\substack{q\sim R \\ q\equiv i \bmod 4}} \frac{S(m_i, n_i; q)}{q^2} e_r(-\epsilon^{-1}\boldsymbol{c}\cdot\boldsymbol{\xi}) I(0) \right| \tag{6.60}$$

to $E_i^{(\text{small})}(w, R)$, where we recall from (6.49) that

$$I(0) = \int_{\mathbb{R}^3} \psi_0(\boldsymbol{x}) e_r\left(\frac{\epsilon\hat{c}_4}{2}\|\boldsymbol{x}\|_2^2 - \boldsymbol{b}\cdot\boldsymbol{x}\right) d\boldsymbol{x}.$$

If $\boldsymbol{c} \neq \boldsymbol{0}$ and $|\hat{c}_4| \leq \frac{1}{100}$ then

$$\|\boldsymbol{b}\|_2^2 = \|\hat{\boldsymbol{c}}\|_2^2 - \hat{c}_4^2 = \|\boldsymbol{c}\|_2^2 - \hat{c}_4^2 \gg 1.$$

It therefore follows from [HBP17, Lemmata 3.1 and 3.2] that

$$I(0) \ll_A \left(\frac{q}{\|\boldsymbol{b}\|_2 Q}\right)^A \ll_A Q^{-\eta A},$$

since $q \leq Q^{1-\eta}$ in this case. The overall contribution to (6.60) from vectors $\boldsymbol{c}$ such that $|\hat{c}_4| \leq \frac{1}{100}$ is therefore seen to be satisfactory.

On interchanging the sum and the integral, we are left with the contribution

$$\ll \frac{\epsilon^5 N^2}{Q^2} \sum_{\substack{\boldsymbol{c}\in\mathcal{C}_i \\ |\hat{c}_4| > \frac{1}{100} \\ (6.59)\text{ holds}}} \int_{[-1,1]^3} |M_i(\boldsymbol{x})| d\boldsymbol{x}, \tag{6.61}$$

where

$$M_i(\boldsymbol{x}) = \sum_{\substack{q\sim R\\ q\equiv i \bmod 4}} \frac{S(m_i,n_i;q)}{q^2} e_r(-\epsilon^{-1}\boldsymbol{c}\cdot\boldsymbol{\xi}) e_r\left(\frac{\epsilon\hat{c}_4}{2}\|\boldsymbol{x}\|_2^2 - \boldsymbol{b}\cdot\boldsymbol{x}\right).$$

But

$$e_r(-\epsilon^{-1}\boldsymbol{c}\cdot\boldsymbol{\xi}) e_r\left(\frac{\epsilon\hat{c}_4}{2}\|\boldsymbol{x}\|_2^2 - \boldsymbol{b}\cdot\boldsymbol{x}\right) = e\left(\frac{2\sqrt{m_i n_i}}{q}\alpha\right),$$

with

$$\begin{aligned}\alpha &= \left(-\epsilon^{-1}\hat{c}_4 + \frac{\epsilon\hat{c}_4\|\boldsymbol{x}\|_2^2}{2} - \boldsymbol{b}\cdot\boldsymbol{x}\right)\cdot\frac{Q}{q}\cdot\frac{q}{2\sqrt{m_i n_i}}\\ &= -\frac{\hat{c}_4}{\|\hat{\boldsymbol{c}}\|_2} + \frac{\epsilon^2\hat{c}_4\|\boldsymbol{x}\|_2^2}{2\|\hat{\boldsymbol{c}}\|_2} - \frac{\epsilon\boldsymbol{b}\cdot\boldsymbol{x}}{\|\hat{\boldsymbol{c}}\|_2}.\end{aligned}$$

But the inequality $\max\{\|\boldsymbol{b}\|_2, |\hat{c}_4|\} \leq \|\hat{\boldsymbol{c}}\|_2$, implies that $|\alpha| \leq 1 + O(\epsilon)$, since $\boldsymbol{x} \in [-1,1]^3$. Thus, it follows from combining partial summation with Conjecture 4.0.2 that $M_i(\boldsymbol{x}) \ll_\delta R^{-1}N^\delta$. (Recall that $\epsilon^{-1} \leq \sqrt{N}$ and $R \leq Q^{1-\eta} \leq Q$.) Returning to (6.61), we conclude that the overall contribution to $E_i^{(\text{small})}(w,R)$ from the main term in Lemma 6.3.6 is

$$\begin{aligned}&\ll_\delta \frac{\epsilon^5 N^{2+\delta}}{RQ^2}\#\left\{\boldsymbol{c}\in\mathcal{C}_i : |\hat{c}_4| > \tfrac{1}{100} \text{ and (6.59) holds}\right\} \ll_\delta \frac{\epsilon^4 N^{2+4\delta}R^\delta}{Q^3}\\ &\qquad\ll_\delta \epsilon N^{\frac{1}{2}+5\delta}.\end{aligned}$$

This is satisfactory for Theorem 6.3.1.

It remains to study the effect of substituting the error term from Lemma 6.3.6 into (6.57). Since $r \leq R/Q \leq Q^{-\eta}$ and $\epsilon|\hat{v}_4| = r^{-1}\epsilon|\hat{c}_4| \ll R^\delta$, by (6.59), we see that the error term is

$$\begin{aligned}&\ll_A \epsilon^3(1+\epsilon|\hat{v}_4|) + \epsilon(1+\epsilon|\hat{v}_4|)^A r^A \ll_A \epsilon^3 R^\delta + \epsilon R^{\delta A}Q^{-\eta A}\\ &\qquad\leq \epsilon^3 R^\delta + \epsilon Q^{A(\delta-\eta)}.\end{aligned}$$

On ensuring that $\delta < \eta$, we see that the second term is an arbitrary negative power of Q and so makes a satisfactory overall contribution to $E_i^{(\text{small})}(w,R)$. In view of (6.56), the contribution from the term $\epsilon^3 N^\delta$ is found to be

$$\ll_\delta \frac{\epsilon^7 N^{2+\delta}}{Q^2 R^{\frac{1}{2}}}\cdot\#\mathcal{C}_i \ll_\delta \frac{\epsilon^7 N^{2+\delta}}{Q^2}\cdot\epsilon^{-1}N^{4\delta} = \frac{\epsilon^6 N^{2+5\delta}}{Q^2}, \tag{6.62}$$

since $R \gg 1$. The right-hand side is $\epsilon^4 N^{1+5\delta}$, which is also satisfactory for Theorem 6.3.1, on redefining δ.

Contribution from small q and large $\epsilon|\hat{v}_4|$

It remains to consider the case $R < Q^{1-\eta}$ and

$$\epsilon|\hat{c}_4| > \frac{R^{1+\delta}}{Q}. \tag{6.63}$$

Let us write $E_i^{(\text{big})}(w, R)$ for the overall contribution to $E_i(w, R)$ from this final case. Our main tool is now Lemma 6.3.7. Let $A \geq 0$. We begin by considering the effect of substituting the main term from this result into (6.57). This yields the contribution

$$\ll \frac{\epsilon^5 N^2}{Q^2} \sum_{\substack{\boldsymbol{c} \in \mathcal{C}_i \\ (6.63) \text{ holds}}} \delta(\hat{\boldsymbol{c}}) \sum_{j=0}^{A} \frac{|k_j|}{(\epsilon|\hat{c}_4|Q)^{\frac{3}{2}+j}} |M_{i,j}|, \tag{6.64}$$

where if $J_{j,q}(\boldsymbol{c})$ is given by (6.53), then

$$M_{i,j} = \sum_{\substack{q \sim R \\ q \equiv i \bmod 4}} \frac{S(m_i, n_i; q)}{q} e_r(-\epsilon^{-1} \boldsymbol{c} \cdot \boldsymbol{\xi}) e_r\left(-\frac{\|\boldsymbol{b}\|_2^2}{2\epsilon \hat{c}_4}\right) q^{\frac{1}{2}+j} J_{j,q}(\boldsymbol{c}).$$

Our plan is to use partial summation to remove the factor $q^{\frac{1}{2}+j} J_{j,q}(\boldsymbol{c})$.

First, as before, we note that

$$e_r(-\epsilon^{-1} \boldsymbol{c} \cdot \boldsymbol{\xi}) e_r\left(-\frac{\|\boldsymbol{b}\|_2^2}{2\epsilon \hat{c}_4}\right) = e\left(\frac{2\sqrt{m_i n_i}}{q} \alpha\right),$$

where

$$\begin{aligned} \alpha &= \left(-\epsilon^{-1} \hat{c}_4 - \frac{\|\boldsymbol{b}\|_2^2}{2\epsilon \hat{c}_4}\right) \cdot \frac{Q}{q} \cdot \frac{q}{2\sqrt{m_i n_i}} \\ &= -\left(\frac{\hat{c}_4}{\|\hat{\boldsymbol{c}}\|_2} + \frac{\|\boldsymbol{b}\|_2^2}{2\hat{c}_4 \|\hat{\boldsymbol{c}}\|_2}\right). \end{aligned}$$

We have $|\alpha| \leq 1 + O(\epsilon^2)$, since $\|\boldsymbol{b}\|_2 \ll \epsilon|\hat{c}_4|$ when $\delta(\hat{\boldsymbol{c}}) \neq 0$. Applying partial summation, based on (6.58) and Lemma 6.3.8, we deduce that

$$M_{i,j} = O_{j,\delta}(R^{\frac{1}{2}+j} N^{3\delta}).$$

Returning to (6.64), we conclude that the overall contribution to $E_i^{(\text{big})}(w, R)$ from the main term in Lemma 6.3.7 is

$$\ll_{\delta,A} \frac{\epsilon^5 N^{2+3\delta}}{Q^2} \sum_{j=0}^{A} \sum_{\substack{\boldsymbol{c} \in \mathcal{C}_i \\ (6.63) \text{ holds}}} \frac{R^{\frac{1}{2}+j}}{(\epsilon|\hat{c}_4|Q)^{\frac{3}{2}+j}} \ll_{\delta,A} \frac{\epsilon^5 N^{2+3\delta}}{Q^2} \cdot \frac{N^{3\delta}}{\epsilon Q} = \epsilon N^{\frac{1}{2}+6\delta}.$$

This is satisfactory for Theorem 6.3.1, on redefining δ.

We must now consider the effect of substituting the error term

$$\ll_A \frac{\epsilon^3}{|\epsilon\hat{v}_4|^{\frac{1}{2}}} + \frac{\epsilon}{|\epsilon\hat{v}_4|^{\frac{5}{2}+A}}$$

from Lemma 6.3.7 into (6.57). Since $q \sim R$, it follows from (6.63) that $\epsilon|\hat{v}_4| \gg R^{\delta}$. The first term is therefore $O(\epsilon^3)$, which makes a satisfactory overall contribution by (6.62). On the other hand, on invoking once more the argument in (6.56), the second term makes the overall contribution

$$\begin{aligned}
&\ll_A \frac{\epsilon^5 N^2}{Q^2} \sum_{\substack{\boldsymbol{c}\in\mathcal{C}_i \\ (6.63) \text{ holds}}} \sum_{q\sim R} \frac{q^{-2}|S(m_i,n_i;q)|}{|\epsilon\hat{v}_4|^{\frac{5}{2}+A}} \\
&\ll_{A,\delta} \frac{\epsilon^5 N^{2+\delta}}{R^{\frac{1}{2}} Q^2} \left(\frac{R}{\epsilon Q}\right)^{\frac{5}{2}+A} \sum_{\substack{\boldsymbol{c}\in\mathcal{C}_i \\ (6.63) \text{ holds}}} \frac{1}{|\hat{c}_4|^{\frac{5}{2}+A}} \\
&\ll_{A,\delta} \frac{\epsilon^4 N^{2+4\delta} R^{\frac{1}{2}-A\delta}}{Q^3}.
\end{aligned}$$

This is $O_\delta(\epsilon N^{\frac{1}{2}+4\delta})$ on assuming that A is is chosen so that $A\delta > \frac{1}{2}$. This is also satisfactory for Theorem 6.3.1, which thereby completes its proof.

6.4 DISCUSSION

Regarding the circle-method approach, it is rather unfortunate that the unconditional result on the twisted Linnik–Selberg Conjecture 4.0.1 is insufficient to show $K(S^3, \mathbb{Z}, 8\mathbb{N}+4) < 2$. The reason for this is that the product of the entries in the Kloosterman sum $N\|\hat{\mathbf{c}}\|_2^2/4$ is rather large. In fact so large, that the $|mn|^{\frac{1}{6}}$-term in Theorem 4.0.1 is by far the dominant term. This term stems from the harder Selberg range, precisely from the transition range of the J-Bessel function, for which it is hard to get further cancellation. One way to get around this is by working with the smooth cut-off already implicit in the circle-method approach. In this case, one might require higher derivatives to deal with the Hankel transforms, which one may derive since the uniform expansions used behave nicely under derivatives. One might imagine using the Laplace or Mellin transform to treat the Hankel transforms. This leads to hypergeometric functions evaluated at points close to the radius of convergence, which may generate further problems.

Alternatively, one may think of a way to adopt the automorphic approach to generate sums of Kloosterman sum to which one may apply the Kuznetsov trace formula. This seems more natural, since the automorphic approach shows that there should be no Maass spectrum. One way to do this is by using the Petersson trace formula. This would be inefficient since we are only interested in the size of a Fourier coefficient of a single cusp form and not the sum of the squares over an orthonormal basis. Nevertheless, due to their relation to the Kloosterman sums it seems that the Poincaré series must be involved in such an argument. One may think of the inner product $\langle G_n, \mathcal{P}^{1,n+2}_{\infty,N}\rangle$ which one may relate to the value of $L(G_n \times \mathcal{P}^{1,n+2}_{\infty,N}, 0)$. One must be careful though as to not argue circularly. A possible approach that has transpired is to apply the Hardy–Ramanujan–Rademacher circle method, that gives rise to an exact formula for the Fourier coefficient, which resembles a formal Dirichlet sum corresponding to $L(G_n \times \mathcal{P}^{1,n+2}_{\infty,N}, 0)$. This venture will be part of future research.

A

APPENDIX: SPECIAL FUNCTIONS AND TRANSFORMS

A.1 WHITTAKER FUNCTION

The Whittaker function as defined as follows [WW96, Section 16.1.2]:

$$\begin{aligned} W_{k,m}(z) &= -\frac{1}{2\pi i}\Gamma\left(k+\frac{1}{2}-m\right)e^{-\frac{1}{2}z}z^k\int_{\infty}^{(0+)}(-t)^{-k-\frac{1}{2}+m}\left(1+\frac{t}{z}\right)^{k-\frac{1}{2}+m}e^{-t}dt \\ &= \frac{e^{-\frac{1}{2}z}z^k}{\Gamma(\frac{1}{2}-k+m)}\int_0^{\infty}t^{-k-\frac{1}{2}+m}\left(1+\frac{t}{z}\right)^{k-\frac{1}{2}+m}e^{-t}dt. \end{aligned} \tag{A.1}$$

In the first formula, the contour is chosen such that the point $-z$ lies outside and the formula is valid for $|\arg(z)| < \pi$, $k, m \in \mathbb{C}$ with $m - k + \frac{1}{2} \notin \mathbb{N}$. The second formula is valid for $|\arg(z)| < \pi$, $k, m \in \mathbb{C}$ with $\mathrm{Re}(m - k + \frac{1}{2}) > 0$.

The Whittaker function satisfies the following recursion formula [DLMF, Eq. 13.15.11]

$$W_{k+1,m}(z) + (2k - z)W_{k,m} + \left((k - \tfrac{1}{2})^2 - m^2\right)W_{k-1,m}(z) = 0, \tag{A.2}$$

and the following relation for its derivative [WW96, Ch. 16 Ex. 3],[DLMF, Eq. 13.15.26]

$$zW'_{k,m}(z) = (\tfrac{z}{2} - k)W_{k,m}(z) - W_{k+1,m}(z). \tag{A.3}$$

The Whittaker function has an elementary expression in the following special case

$$W_{k,\pm(k-\frac{1}{2})}(z) = e^{-\frac{1}{2}z}z^k. \tag{A.4}$$

This can be easily seen from (A.1) using the residue theorem and is also recorded in [DLMF, Eq. 13.18.2]. We shall further require the following integral involving the Whittaker function [EMOT81, Section 20.3, Eq. (30)], [DLMF, Eq. 13.23.4]

$$\int_0^{\infty} e^{-\frac{z}{2}}W_{\frac{\kappa}{2},it}(z)z^{s-1}dz = \frac{\Gamma(s+\frac{1}{2}-it)\Gamma(s+\frac{1}{2}+it)}{\Gamma(s+1-\frac{\kappa}{2})}, \quad \forall\,\mathrm{Re}(s) > |\mathrm{Re}(it)| - \frac{1}{2}. \tag{A.5}$$

A.2 BESSEL FUNCTIONS

For an extensive treatment of Bessel functions we refer to [Wat44]. Here, we recollect the basic definitions, some useful integral and series representations, and some asymptotic behaviours. The Bessel function of the first kind is given by

$$J_v(z) = \frac{(\frac{1}{2}z)^k}{2\pi i} \int_{-\infty}^{(0+)} t^{-v} \exp\left(t - \frac{z^2}{4t}\right) \frac{dt}{t}. \tag{A.6}$$

The Bessel function of the second kind is given by

$$Y_v(z) = \frac{J_v(z)\cos(v\pi) - J_{-v}(z)}{\sin(v\pi)}, \tag{A.7}$$

where for $v \in \mathbb{Z}$ this is to be regarded as a limit. The modified Bessel function of the first kind is given by

$$I_v(z) = \begin{cases} e^{-\frac{1}{2}\pi i v} J_v(z e^{\frac{\pi i}{2}}), & -\pi < \arg z \leq \frac{\pi}{2}, \\ e^{\frac{3}{2}\pi i v} J_v(z e^{-\frac{3}{2}\pi i}), & \frac{\pi}{2} < \arg z \leq \pi. \end{cases} \tag{A.8}$$

This defintion becomes more natural when comparing the series representations (A.10) and (A.14). The modified Bessel function of the second kind is given by

$$K_v(z) = \frac{\pi}{2} \frac{I_{-v}(z) - I_v(z)}{\sin(v\pi)}, \tag{A.9}$$

where for $v \in \mathbb{Z}$ this is to be regarded as a limit.

We shall require some integral and series representations of Bessel functions:

$$J_v(z) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{z}{2}\right)^{v+2m}}{m!\Gamma(v+m+1)}, \quad \forall v, z \in \mathbb{C}, \tag{A.10}$$

$$J_v(z) = \frac{1}{\pi} \int_0^{\pi} \cos(v\theta - z\sin(\theta)) d\theta - \frac{\sin(v\pi)}{\pi} \int_0^{\infty} e^{-v\theta - z\sinh(\theta)} d\theta,$$
$$\forall v, z \in \mathbb{C}, \mathrm{Re}(z) > 0, \tag{A.11}$$

$$J_v(x) = \frac{2}{\pi} \int_0^{\infty} \sin\left(x\cosh(\xi) - \frac{\pi}{2}v\right) \cosh(v\xi) d\xi,$$
$$\forall v \in \mathbb{C}, -1 < \mathrm{Re}(v) < 1, \forall x \in \mathbb{R}^+, \tag{A.12}$$

$$Y_v(x) = -\frac{2\left(\frac{x}{2}\right)^{-v}}{\sqrt{\pi}\Gamma(\frac{1}{2}-v)} \int_1^\infty \frac{\cos(xt)}{(t^2-1)^{v+\frac{1}{2}}} dt,$$
$$\forall x \in \mathbb{R}^+, \forall v \in \mathbb{C}, -\frac{1}{2} < \operatorname{Re}(v) < \frac{1}{2}, \quad \text{(A.13)}$$

$$I_v(z) = \sum_{m=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{v+2m}}{m!\Gamma(v+m+1)}, \quad \forall v, z \in \mathbb{C}, \quad \text{(A.14)}$$

$$K_v(az) = \frac{1}{2}a^v \int_0^\infty \exp\left(-\frac{1}{2}z\left(t+\frac{a^2}{t}\right)\right) \frac{dt}{t^{v+1}},$$
$$\forall a, v, z \in \mathbb{C}, \operatorname{Re}(z) > 0, \operatorname{Re}(a^2 z) > 0, \quad \text{(A.15)}$$

$$K_v(z) = \frac{\Gamma(\frac{1}{2}+v)(2z)^v}{\sqrt{\pi}} \int_0^\infty \frac{\cos(t)}{(t^2+z^2)^{\frac{1}{2}+v}} dt,$$
$$\forall z, v \in \mathbb{C}, \operatorname{Re}(v) > -\tfrac{1}{2}, \operatorname{Re}(z) > 0. \quad \text{(A.16)}$$

$$K_v(x) = \frac{1}{2\pi i} \int_{(\sigma)} 2^{s-1} x^{-s} \Gamma\left(\frac{s+v}{2}\right) \Gamma\left(\frac{s-v}{2}\right) ds,$$
$$\forall v \in \mathbb{C}, \forall x, \sigma \in \mathbb{R}^+, \sigma > |\operatorname{Re}(v)|. \quad \text{(A.17)}$$

All of the above can be found in [Wat44] with the exception of the last equation, which follows from Mellin inversion and the Mellin transform of $K_v(x)$, which can be found in [Iwa02, Appendix B].

We further also require some asymptotic expansions of Bessel functions, which can be found in [Iwa02, Appendix B]:

$$J_v(x) = \sqrt{\frac{2}{\pi x}} \cos\left(x - \tfrac{\pi}{2}v - \tfrac{\pi}{4}\right) + O\left(\frac{1+|v|^2}{x^{\frac{3}{2}}}\right), \quad \forall v \in \mathbb{C}, \forall x \in \mathbb{R}^+, \quad \text{(A.18)}$$

$$Y_v(x) = \sqrt{\frac{2}{\pi x}} \sin\left(x - \tfrac{\pi}{2}v - \tfrac{\pi}{4}\right) + O\left(\frac{1+|v|^2}{x^{\frac{3}{2}}}\right), \quad \forall v \in \mathbb{C}, \forall x \in \mathbb{R}^+, \quad \text{(A.19)}$$

A.3 SPECIAL INTEGRALS

$$\int_{-\infty}^{\infty}(1-iu)^{-s_1}(1+iu)^{-s_2}du = \pi 2^{2-s_1-s_2}\frac{\Gamma(s_1+s_2-1)}{\Gamma(s_1)\Gamma(s_2)},$$
$$\forall s_1, s_2 \in \mathbb{C}, \mathrm{Re}(s_1), \mathrm{Re}(s_2) > 0, \mathrm{Re}(s_1+s_2) > 1. \quad \text{(A.20)}$$

See [GR07, page 909 eq. 8.381.1.].

A.4 TRANSFORMS

Proposition A.4.1 (Fourier Transform). *Let $\phi \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R})$ with $\phi, \phi', \phi'' \in L^1(\mathbb{R}^n)$. Then, we have*

$$\phi(\boldsymbol{x}) = \int_{\mathbb{R}^n} \mathcal{F}[\phi](\boldsymbol{\xi}) e(\boldsymbol{x}\cdot\boldsymbol{\xi}) d\boldsymbol{\xi},$$

where

$$\mathcal{F}[\phi](\boldsymbol{\xi}) = \int_{\mathbb{R}^n} \phi(\boldsymbol{x}) e(-\boldsymbol{x}\cdot\boldsymbol{\xi}) d\boldsymbol{x}$$

is the Fourier transform.

Proof. See [IK04, Section 4.A]. □

Proposition A.4.2 (Mellin Transform). *Let $\phi \in \mathcal{C}^2(\mathbb{R}^+, \mathbb{R})$ with*

$$\phi(x)x^{s-1}, \phi'(x)x^s, \phi''(x)x^{s+1} \in L^1(\mathbb{R}^+) \text{ for } \alpha < \mathrm{Re}(s) < \beta.$$

Then, we have

$$\phi(x) = \frac{1}{2\pi i}\int_{(\sigma)} \mathcal{M}[\phi](s) x^{-s} ds,$$

for any $\sigma \in]\alpha, \beta[$, where

$$\mathcal{M}[\phi](s) = \int_0^{\infty} \phi(x) x^{s-1} dx, \quad \forall s \in \mathbb{C} : \mathrm{Re}(s) \in]\alpha, \beta[,$$

is the Mellin transform.

Proof. See [IK04, Section 4.A]. □

Proposition A.4.3 (Kontorovitch–Lebedev Inversion). *Let $\phi \in \mathcal{C}^2(\mathbb{R}_0^+, \mathbb{R})$ with $\phi(0) = 0$ and $\phi(x), \phi'(x), \phi''(x) \ll (x+1)^{-2+\delta}$ for some $\delta > 0$. Then, we have*

$$\phi(x) = \frac{2}{\pi}\int_{-\infty}^{\infty} K_{2it}(x)\check{\phi}(t)\sinh(\pi t) t dt,$$

where

$$\check{\phi}(t) = \frac{4}{\pi}\cosh(\pi t)\int_0^\infty K_{2it}(x)\phi(x)\frac{dx}{x}.$$

Proof. See [KL38, KL39] and [Leb72, p. 131]. □

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